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Bilateral Hardy-type Inequalities

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Abstract

This paper studies the Hardy-type inequalities on the intervals (may be infinite) with two weights, either vanishing at two endpoints of the interval or having mean zero. For the first type of inequalities, in terms of new isoperimetric constants, the factor of upper and lower bounds becomes smaller than the known ones. The second type of the inequalities is motivated from probability theory and is new in the analytic context. The proofs are now rather elementary. Similar improvements are made for Nash inequality, Sobolev-type inequality, and the logarithmic Sobolev inequality on the intervals.

A large number of results on Hardy-type inequalities have been already collected and explored in the books [10] – [12], [16], and [18]. This paper makes two additions. The first one is for the functions vanishing at two endpoints of the interval. This type of inequalities was included in [18]. The contribution here is some improvement, not only on the isoperimetric constant but also on the factor of the upper and lower bounds. The second addition is for the case where the functions have mean zero, which is motivated from a probabilistic consideration and is not included in the books cited above. These two cases are studied in the next two sections separately. The main result in each case is stated as a theorem (Theorems 1.6 and 2.6). Their extensions to more general setup are presented as Theorems 1.11 and 2.9. As applications of the results or ideas developed in the first two section, in the third section, we study the Nash inequality, the Sobolev-type inequality, and the logarithmic Sobolev inequality. The paper can be regarded as an extension of the L^2 -case studied in [6, 7].

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1 The case of vanishing boundaries.

Consider an interval $[-M, N]$ with $M, N \leq \infty$. Certainly, here $[-M, N]$ means $[-M, N)$ if $N = \infty$. This costs no confusion. In this section, we study the case that the functions vanish at two endpoints of the interval. That is the Hardy-type inequality:

$$\left(\int_{-M}^N |f|^q d\mu \right)^{1/q} \leq A \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p}, \quad f(-M)=0 \text{ and } f(N)=0, \quad (1)$$

where $f(N) = 0$ for instance means that $\lim_{x \rightarrow \infty} f(x) = 0$ if $N = \infty$. Throughout the paper, all of the functions involved in the Hardy-type inequalities are assumed to be absolutely continuous without mentioned time by time in what follows. In the special case that $p = q = 2$, the results in this and the next sections are proved in [6, 7]) using much advanced methods. The present study is motivated from seeking for more direct proofs for the results. At the moment, it is unclear how the capacitary technique used in [6, 7] can be applied in the present general setup. It may be helpful to the reader by studying the problem step by step to show how to find out the main result. The study consists of five steps. At each step, we have either a proposition or a lemma. If one is in hurry, who may jump from here to the main results, Theorems 1.11 and 2.9.

Our first step is using the splitting technique (which we have used several times before, cf. [8], [3; Theorems 3.3 and 3.4], and [5]). To do so, fix $\theta \in (-M, N)$ and denote by A_θ^+ and A_θ^- , respectively, the optimal constant in the following inequalities.

$$\begin{aligned} \left(\int_\theta^N |f|^q d\mu \right)^{1/q} &\leq A_\theta^+ \left(\int_\theta^N |f'|^p d\nu \right)^{1/p}, & f(N) = 0, \\ \left(\int_{-M}^\theta |f|^q d\mu \right)^{1/q} &\leq A_\theta^- \left(\int_{-M}^\theta |f'|^p d\nu \right)^{1/p}, & f(-M) = 0. \end{aligned}$$

Clearly, these inequalities are different from (1) since only one-side boundary condition is endowed. Here and in what follows the superscript “−” means on the left-hand side of θ and “+” means on the right-hand side of θ .

The next result shows that we can describe the optimal constant A in (1) in terms of A_θ^\pm which are the optimal constants on half-spaces with different boundary conditions.

Proposition 1.1 For $1 \leq p \leq q < \infty$, we have

$$2^{1/q-1/p} \sup_{\theta \in [-M, N]} (A_\theta^- \wedge A_\theta^+) \leq A \leq \inf_{\theta \in [-M, N]} (A_\theta^- \vee A_\theta^+),$$

where $A_N^+ = 0$ and $A_{-M}^- = 0$ by convention, $\alpha \wedge \beta = \min\{\alpha, \beta\}$, and $\alpha \vee \beta = \max\{\alpha, \beta\}$.

Proof. (a) For each fixed $\theta \in [-M, N]$ and f with $f(-M) = 0$ and $f(N) = 0$, by the inequalities on the half-spaces, we have

$$\begin{aligned}
\int_{-M}^N |f'|^p d\nu &= \int_{-M}^{\theta} |f'|^p d\nu + \int_{\theta}^N |f'|^p d\nu \\
&\geq (A_{\theta}^{-})^{-p} \left(\int_{-M}^{\theta} |f|^q d\mu \right)^{p/q} + (A_{\theta}^{+})^{-p} \left(\int_{\theta}^N |f|^q d\mu \right)^{p/q} \\
&\geq \left[(A_{\theta}^{-})^{-p} \wedge (A_{\theta}^{+})^{-p} \right] \left[\left(\int_{-M}^{\theta} |f|^q d\mu \right)^{p/q} + \left(\int_{\theta}^N |f|^q d\mu \right)^{p/q} \right] \\
&\geq (2^{(p/q-1) \vee 0})^{-1} \left[(A_{\theta}^{-})^{-p} \wedge (A_{\theta}^{+})^{-p} \right] \left(\int_{-M}^N |f|^q d\mu \right)^{p/q} \\
&\quad \text{(by } c_r\text{-inequality).}
\end{aligned}$$

Since f is arbitrary, we have

$$A^p \leq 2^{(p/q-1) \vee 0} \left[(A_{\theta}^{-})^p \vee (A_{\theta}^{+})^p \right].$$

Now, since θ is arbitrary, we obtain

$$A \leq 2^{(1/q-1/p) \vee 0} \inf_{\theta \in [-M, N]} (A_{\theta}^{-} \vee A_{\theta}^{+}).$$

This conclusion holds for general $p, q \in [1, \infty)$.

(b) Again, fix θ . Suppose for a moment that we can construct two absolutely continuous functions f_- and f_+ having the following properties: $f_-(-M) = 0$, $f'_-(\theta) = 0$, $f_-(\theta) > 0$,

$$\int_{-M}^{\theta} |f_-|^q d\mu = 1, \quad \text{and} \quad \left(\int_{-M}^{\theta} |f'_-|^p d\nu \right)^{1/p} < (A_{\theta}^{-})^{-1} + \varepsilon;$$

$f_+(N) = 0$, $f'_+(\theta) = 0$, $f_+(\theta) > 0$,

$$\int_{\theta}^N |f_+|^q d\mu = 1, \quad \text{and} \quad \left(\int_{\theta}^N |f'_+|^p d\nu \right)^{1/p} < (A_{\theta}^{+})^{-1} + \varepsilon.$$

Set $f = cf_- \mathbb{1}_{[-M, \theta]} + f_+ \mathbb{1}_{(\theta, N]}$, where $c = f_+(\theta)/f_-(\theta)$. Then

$$\begin{aligned}
1 + |c|^q &= \int_{-M}^{\theta} |cf_-|^q d\mu + \int_{\theta}^N |f_+|^q d\mu = \int_{-M}^N |f|^q d\mu, \\
\int_{-M}^N |f'|^p d\nu &= |c|^p \int_{-M}^{\theta} |f'_-|^p d\nu + \int_{\theta}^N |f'_+|^p d\nu \\
&\leq |c|^p \left((A_{\theta}^{-})^{-1} + \varepsilon \right)^p + \left((A_{\theta}^{+})^{-1} + \varepsilon \right)^p \\
&\leq \left((A_{\theta}^{-})^{-1} \vee (A_{\theta}^{+})^{-1} + \varepsilon \right)^p (1 + |c|^p).
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\int_{-M}^N |f'|^p d\nu \right)^{1/p} &\leq \left((A_\theta^-)^{-1} \vee (A_\theta^+)^{-1} + \varepsilon \right) (1 + |c|^p)^{1/p} \\
&\leq 2^{1/p-1/q} \left((A_\theta^-)^{-1} \vee (A_\theta^+)^{-1} + \varepsilon \right) (1 + |c|^q)^{1/q} \\
&\quad \text{(by Jensen's inequality requiring } q \geq p \text{)} \\
&= 2^{1/p-1/q} \left((A_\theta^-)^{-1} \vee (A_\theta^+)^{-1} + \varepsilon \right) \left(\int_{-M}^N |f|^q d\mu \right)^{1/q}.
\end{aligned}$$

Thus, whenever $q \geq p$, we have

$$A \geq 2^{1/q-1/p} (A_\theta^- \wedge A_\theta^+).$$

Therefore, we obtain

$$A \geq 2^{1/q-1/p} \sup_{\theta \in [-M, N]} (A_\theta^- \wedge A_\theta^+), \quad 1 \leq p \leq q < \infty.$$

Combining this with (a), we arrive at the conclusion of the proposition.

(c) To complete the proof, it remains to construct the functions f_- and f_+ used in (b). For this, we need consider f_- only by symmetry. The problem is the condition at θ : $f'_-(\theta) = 0$ and $f_-(\theta) > 0$. The proof given below is modified from [9; Proof (ii) of Theorem 1.1]. If necessary, by modifying f_- properly on a sufficiently small neighborhood of θ , we can assume that $f'_-(\theta) = 0$. The main point here is to modify f_- so that we also have $f_-(\theta) \neq 0$. Otherwise, suppose that $f_-(\theta) = 0$. Since f_- is absolutely continuous, $f_-(-M) = 0$ and $f_-(\theta) = 0$, there exists $x_1 \in (-M, \theta)$ such that $|f_-(x_1)| = \sup_{x \in (-M, \theta)} |f_-(x)|$. Then $f_-(x_1) \neq 0$ (otherwise, $f \equiv 0$ which contradicts with the norm 1 assumption). Let $\tilde{f}_- = f_- \mathbb{1}_{[-M, x_1]} + f_-(x_1) \mathbb{1}_{[x_1, \theta]}$. Then \tilde{f}_- is absolutely continuous,

$$\begin{aligned}
c^q &:= \int_{-M}^\theta |\tilde{f}_-|^q d\mu \geq \int_{-M}^\theta |f_-|^q d\mu = 1, \\
\left(\int_{-M}^\theta |\tilde{f}_-|^p d\nu \right)^{1/p} &\leq \left(\int_{-M}^\theta |f'_-|^p d\nu \right)^{1/p} < (A_\theta^-)^{-1} + \varepsilon.
\end{aligned}$$

Set $\bar{f}_- = c\tilde{f}_-$. Now it follows that

$$\bar{f}_-(-M) = 0, \quad \bar{f}'_-(\theta) = 0, \quad \bar{f}_-(\theta) \neq 0, \quad \int_{-M}^\theta |\bar{f}_-|^q d\mu = 1,$$

and

$$\left(\int_{-M}^\theta |\bar{f}'_-|^p d\nu \right)^{1/p} = \frac{1}{c} \left(\int_{-M}^\theta |\tilde{f}'_-|^p d\nu \right)^{1/p} < (A_\theta^-)^{-1} + \varepsilon.$$

Hence, we can replace f_- by \bar{f}_- when $f_-(\theta) = 0$. \square

Having Proposition 1.1 at hand, it is ready to write down some estimates of the optimal constant A in (1), as we did in [3, 5], in terms of B_θ^\pm given below (cf. [18; Theorem 6.2] and [16; §1.3, Theorem 3] in which the factor $k_{q,p}$ may be different):

$$B_\theta^\pm \leq A_\theta^\pm \leq k_{q,p} B_\theta^\pm, \quad 1 < p \leq q < \infty, \quad (2)$$

$$B_\theta^+ = \sup_{r \in (\theta, N)} \mu[\theta, r]^{1/q} \left[\int_r^N \left(\frac{d\nu^*}{dx} \right)^{-1/(p-1)} dx \right]^{(p-1)/p}, \quad (3)$$

$$B_\theta^- = \sup_{r \in (-M, \theta)} \mu[r, \theta]^{1/q} \left[\int_{-M}^r \left(\frac{d\nu^*}{dx} \right)^{-1/(p-1)} dx \right]^{(p-1)/p}, \quad (4)$$

where ν^* is the absolutely continuous part of ν and $k_{q,p}$ is a universal constant will be used often in this paper:

$$k_{q,p} = \left(1 + \frac{q}{p'} \right)^{1/q} \left(1 + \frac{p'}{q} \right)^{1/p'}, \quad (5)$$

where p' is the conjugate number of p and similarly for q' . On the half-line when $q > p$, the constant is improved as follows:

$$k_{q,p} = \left[\frac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(\frac{p(q-1)}{q-p})} \right]^{1/p-1/q} = \left[\frac{q-p}{pqB(\frac{q}{q-p}, \frac{p(q-1)}{q-p})} \right]^{1/p-1/q}, \quad q > p,$$

where $\Gamma(x)$ and $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ are Gamma and Beta functions, respectively (cf. [2; Theorem 8], [13; Theorem 2], and also [11; pages 45–47] for historical remarks). According to Lebesgue's decomposition theorem, each measure ν can be decomposed into three parts:

$$\nu = \nu^* + \nu_{\text{sing}} + \nu_{\text{pp}},$$

where ν_{sing} is the singular continuous part and ν_{pp} is the pure point part (a discrete measure).

We are now going to present some more explicit estimates. To do so, we need the following simple result.

Lemma 1.2 For a given Borel measure μ and positive functions φ and ψ on $[-M, N]$, we have

$$\sup_{(x,y): x \leq y} \frac{\mu[x, y]}{\varphi(x) + \psi(y)} \geq \sup_{\theta} \left\{ \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\varphi(x)} \right] \bigwedge \left[\sup_{y \geq \theta} \frac{\mu(\theta, y]}{\psi(y)} \right] \right\}.$$

Proof. For fixed $x \leq y$ and $(x, y) \ni \theta$, we have by proportional property that

$$\frac{\mu[x, y]}{\varphi(x) + \psi(y)} = \frac{\mu[x, \theta] + \mu(\theta, y]}{\varphi(x) + \psi(y)} \geq \frac{\mu[x, \theta]}{\varphi(x)} \bigwedge \frac{\mu(\theta, y]}{\psi(y)}$$

and furthermore

$$\frac{\mu[x, y]}{\varphi(x) + \psi(y)} \geq \sup_{\theta \in [x, y]} \left\{ \frac{\mu[x, \theta]}{\varphi(x)} \wedge \frac{\mu(\theta, y)}{\psi(y)} \right\}.$$

Thus,

$$\begin{aligned} \sup_{x \leq y} \frac{\mu[x, y]}{\varphi(x) + \psi(y)} &\geq \sup_{x \leq y} \sup_{\theta \in [x, y]} \left\{ \frac{\mu[x, \theta]}{\varphi(x)} \wedge \frac{\mu(\theta, y)}{\psi(y)} \right\} \\ &= \sup_{\theta} \sup_{[x, y] \ni \theta} \{ \dots \} \\ &= \sup_{\theta} \left\{ \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\varphi(x)} \right] \wedge \left[\sup_{y \geq \theta} \frac{\mu(\theta, y)}{\psi(y)} \right] \right\} \end{aligned}$$

as required. \square

It is remarkable that we do not have an expected dual result of the above lemma. At beginning, we do have the dual

$$\frac{\mu[x, y]}{\varphi(x) + \psi(y)} \leq \frac{\mu[x, \theta] + \mu(\theta, y)}{\varphi(x) + \psi(y)} \leq \frac{\mu[x, \theta]}{\varphi(x)} \vee \frac{\mu(\theta, y)}{\psi(y)}.$$

Hence

$$\sup_{x \leq \theta \leq y} \frac{\mu[x, y]}{\varphi(x) + \psi(y)} \leq \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\varphi(x)} \right] \vee \left[\sup_{y \geq \theta} \frac{\mu(\theta, y)}{\psi(y)} \right]$$

and furthermore

$$\inf_{\theta} \sup_{x \leq \theta \leq y} \frac{\mu[x, y]}{\varphi(x) + \psi(y)} \leq \inf_{\theta} \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\varphi(x)} \right] \vee \left[\sup_{y \geq \theta} \frac{\mu(\theta, y)}{\psi(y)} \right].$$

Clearly, this is somehow a dual of Lemma 1.2 but it is still a distance to what we expect:

$$\sup_{x \leq y} \frac{\mu[x, y]}{\varphi(x) + \psi(y)} \leq \inf_{\theta} \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\varphi(x)} \right] \vee \left[\sup_{y \geq \theta} \frac{\mu(\theta, y)}{\psi(y)} \right].$$

Alternatively, let $\bar{\theta}$ satisfy

$$\sup_{x \leq \bar{\theta}} \frac{\mu[x, \bar{\theta}]}{\varphi(x)} = \sup_{y \geq \bar{\theta}} \frac{\mu(\bar{\theta}, y)}{\psi(y)}.$$

Then we have

$$\sup_{x \leq \bar{\theta} \leq y} \frac{\mu[x, y]}{\varphi(x) + \psi(y)} \leq \sup_{x \leq \bar{\theta}} \frac{\mu[x, \bar{\theta}]}{\varphi(x)} = \left[\sup_{x \leq \bar{\theta}} \frac{\mu[x, \bar{\theta}]}{\varphi(x)} \right] \vee \left[\sup_{y \geq \bar{\theta}} \frac{\mu(\bar{\theta}, y)}{\psi(y)} \right]. \quad (6)$$

Very often, the right-hand side coincides with

$$\inf_{\theta} \left\{ \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\varphi(x)} \right] \vee \left[\sup_{y \geq \theta} \frac{\mu(\theta, y)}{\psi(y)} \right] \right\},$$

but one can not remove $\bar{\theta}$ from the left-hand side and keep the inequality.

Throughout this paper, we mainly restrict ourselves to the case that $1 < p \leq q < \infty$. The limit case that either $p = 1$ or $q = \infty$ are easier and so are omitted here. For simplicity, throughout this paper, we set

$$h(x) = \left(\frac{d\nu^*}{dx} \right)^{-1/(p-1)}, \quad \hat{\nu}(dx) = h(x)dx.$$

Clearly, h and $\hat{\nu}$ depend on $p > 1$. The measure $\hat{\nu}$ comes, but different, from ν . In what follows, almost every estimate is expressed by using the pair $(\mu, \hat{\nu})$ but not (μ, ν) . Besides, we may assume that

$$\hat{\nu}(-M, N) := \int_{-M}^N h = \int_{-M}^N \left(\frac{d\nu^*}{dx} \right)^{-1/(p-1)} dx < \infty. \quad (7)$$

This technical assumption can often be avoided by replacing $d\nu^*/dx$ with $d\nu^*/dx + \varepsilon \exp[(p-1)x^2]$ and then passing to the limit as $\varepsilon \downarrow 0$. Alternatively, one may start at $M, N < \infty$, replace $d\nu^*/dx$ with $d\nu^*/dx + \varepsilon$. Then pass to the limit as $\varepsilon \downarrow 0$, and then as $M, N \rightarrow \infty$ if necessary. In parallel, without loss of generality, we can also assume that μ is positive on each subinterval.

Next, define a constant B^* by

$$(B^{*q})^{-1} = \inf_{-M \leq x \leq y \leq N} \left[\hat{\nu}[-M, x]^{-\frac{q(p-1)}{p}} + \hat{\nu}[y, N]^{-\frac{q(p-1)}{p}} \right] \mu[x, y]^{-1}. \quad (8)$$

Let us now discuss the boundary condition in the definition of B^* above (or B_* below), when $M = \infty$, here $x = -M$ means that $x \rightarrow -\infty$:

$$\begin{aligned} & \overline{\lim}_{x \rightarrow -\infty} \mu[x, y]^{1/q} \left[\hat{\nu}[-M, x]^{-\frac{q(p-1)}{p}} + \hat{\nu}[y, N]^{-\frac{q(p-1)}{p}} \right]^{-1/q} \\ &= \overline{\lim}_{x \rightarrow -\infty} \mu[x, y]^{1/q} \hat{\nu}[-M, x]^{(p-1)/p} \end{aligned}$$

which is the type $\infty \cdot 0$ of limit provided $\mu[-\infty, y] = \infty$. Otherwise, the limit is zero and so the boundary $-M$ can be ignored in computing B^* . When $M = \infty = N$, we need to compute the iterated limit only. To which, the main reason is that the optimal constant A is increasing as either $N \uparrow$ or $-M \downarrow$. Hence, the general case can be regarded as the limit of finite M and N . In other words, we do not need to consider the other types of double limits as $M, N \rightarrow \infty$.

Here is our upper estimate.

Lemma 1.3 Let $\mu_{pp} = 0$. Then for $1 < p \leq q < \infty$, we have $A \leq k_{q,p} B^*$, where B^* is defined by (8).

Proof. As mentioned above, without loss of generality, we can assume (7). Rewrite B^{*q} as

$$B^{*q} = \sup_{x \leq y} \frac{\mu[x, \theta] + \mu(\theta, y]}{\hat{\nu}[-M, x]^{-q(p-1)/p} + \hat{\nu}[y, N]^{-q(p-1)/p}}.$$

As an application of Lemma 1.2, we have

$$\begin{aligned} B^{*q} &\geq \sup_{\theta} \left\{ \left[\sup_{x \leq \theta} \frac{\mu[x, \theta]}{\hat{\nu}[-M, x]^{-q(p-1)/p}} \right] \wedge \left[\sup_{y \geq \theta} \frac{\mu(\theta, y]}{\hat{\nu}[y, N]^{-q(p-1)/p}} \right] \right\} \\ &= \sup_{\theta} [B_{\theta}^{-} \wedge B_{\theta}^{+}]^q. \end{aligned}$$

Here in the last step, we have used the condition $\mu_{pp} = 0$. Since we can represent

$$\mu_{\text{sing}}[x, \theta] = \mu_{\text{sing}}[-M, \theta] - \mu_{\text{sing}}[-M, x]$$

(the last term is continuous in x) and $\mu_{pp} = 0$, it follows that the function $\mu[x, \theta]$ is continuous in x and θ . By choosing $\bar{\theta}$ such that $B_{\bar{\theta}}^{-} = B_{\bar{\theta}}^{+}$, it follows that $B_{\bar{\theta}}^{-} \leq B^{*}$ (just proved) and furthermore

$$\begin{aligned} A &\leq \inf_{\theta \in [-M, N]} (A_{\theta}^{-} \vee A_{\theta}^{+}) \quad (\text{by Proposition 1.1}) \\ &\leq k_{q,p} \inf_{\theta \in [-M, N]} (B_{\theta}^{-} \vee B_{\theta}^{+}) \quad (\text{by (2)}) \\ &\leq k_{q,p} B_{\bar{\theta}}^{-} \quad (\text{by definition of } \bar{\theta}) \\ &\leq k_{q,p} B^{*}. \quad \square \end{aligned}$$

Note that the parameter θ is used temporary in the proof above. Thus, the splitting procedure is a bridge to go to the upper estimate but our final result does not depend on the splitting points. The simple technique used in proving the upper estimate above is in common, and will be used several times later (Lemma 2.3 and Theorem 3.2).

Before moving further, let us discuss the technical assumption that $\mu_{pp} = 0$. In the study of B_{θ}^{\pm} for half-spaces, one first handles with the case that $\mu \ll dx$ and $\nu \ll dx$ and then removes this restriction by the following technique. Without loss of generality, assume that $M, N < \infty$. Besides, we may also assume that $f' \geq 0$ in the study of the upper estimate. The idea is to use an approximating procedure (cf. [17] or [16; page 45]). Note that

$$\left(\int_{-M}^N f^q d\mu \right)^{1/q} = \left(\int_{-M}^N \mu[x, N] df(x)^q \right)^{1/q}.$$

Now, we can approximate $\mu[x, N]$ by a sequence of absolutely continuous, decreasing functions $\{g_n\}$ having the property: $g_n \leq \mu[\cdot, N]$ for every n ; as $n \rightarrow \infty$, $g_n(x)$ converges to $\mu[x, N]$ for almost all x . Thus, we can first replace

$\mu[x, N]$ by absolutely continuous g_n and then pass to the limit as $n \rightarrow \infty$. Actually, now $\mu[\cdot, N]$ consists of three parts: the absolutely continuous part, the singular continuous one plus a step function. Each of them is decreasing. There is nothing to do about the absolutely continuous part. The singular decreasing continuous function can be approximated from below by decreasing step functions. Furthermore, each of the step functions can be approximated from below almost everywhere by absolutely continuous decreasing functions. The new difficulty arises: even though we have the control $g_n(x) \leq \mu[x, N]$ for all x , but we still do not know how to construct a sequence $\{g_n\}$ as above having the control $(0 \leq) g_n(x) - g_n(y) \leq \mu[x, y]$ for every pair $\{x, y\}$ with $x < y$ and each n .

For the lower bound of A , the dual proof of Lemma 1.2 does not work well as remarked below Lemma 1.2. More precisely, what we can obtain by Proposition 1.1 and (6) is as follows.

$$A \geq 2^{1/q-1/p} \sup_{x \leq \bar{\theta} \leq y} \left\{ \mu[x, y]^{1/q} \left[\hat{\nu}[-M, x]^{-\frac{q(p-1)}{p}} + \hat{\nu}[y, N]^{-\frac{q(p-1)}{p}} \right]^{-1/q} \right\}, \quad (9)$$

where $\bar{\theta}$ is the solution of the equation $B_{\bar{\theta}}^- = B_{\bar{\theta}}^+$. The result is less satisfactory since $\bar{\theta}$ (unknown explicitly) is included. Fortunately, there is a direct technique (cf. [6; Proof (b) of Theorem 8.2]) to handle with the lower estimate.

Lemma 1.4 For $1 < p, q < \infty$, we have

$$A \geq \sup_{-M \leq x \leq y \leq N} \left\{ \mu[x, y]^{1/q} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-1/p} \right\} =: B_*.$$

Proof. Given $m', m, \theta, n, n' \in [-M, N]$ with $m' < m < \theta < n < n'$, define

$$f(x) = \gamma \mathbb{1}_{\{m' \leq x \leq \theta\}} \hat{\nu}[m', x \wedge m] + \mathbb{1}_{\{\theta < x \leq n'\}} \hat{\nu}[x \vee n, n']$$

where $\gamma = \hat{\nu}[n, n'] / \hat{\nu}[m', m]$. Clearly, f is absolutely continuous. We have

$$\left(\int_{-M}^N |f|^q d\mu \right)^{1/q} = \left(\int_{m'}^{n'} |f|^q d\mu \right)^{1/q} \geq \left(\int_m^{n'} |f|^q d\mu \right)^{1/q} = \mu[m, n]^{1/q} \hat{\nu}[n, n']$$

and

$$\begin{aligned} \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p} &= \left(\gamma^p \int_{m'}^m h^p d\nu + \int_n^{n'} h^p d\nu \right)^{1/p} \\ &= (\gamma^p \hat{\nu}[m', m] + \hat{\nu}[n, n'])^{1/p}. \end{aligned}$$

Here in the last step, we have ignored the singular part of ν since the original inequality is equivalent to the one having $\nu = \nu^*$. To see this, simply set $f' = 0$ on the singular part of ν . Thus, the optimal constant A satisfies

$$A \geq (\mu[m, n]^{1/q} \hat{\nu}[n, n']) (\gamma^p \hat{\nu}[m', m] + \hat{\nu}[n, n'])^{-1/p}.$$

But

$$\begin{aligned} & (\gamma^p \hat{\nu}[m', m] + \hat{\nu}[n, n']) \hat{\nu}[n, n']^{-p} \\ &= \{ \hat{\nu}[n, n']^p \hat{\nu}[m', m]^{-p} \hat{\nu}[m', m] + \hat{\nu}[n, n'] \} \hat{\nu}[n, n']^{-p} \\ &= \hat{\nu}[m', m]^{1-p} + \hat{\nu}[n, n']^{1-p}, \end{aligned}$$

it follows that

$$A \geq \mu[m, n]^{1/q} (\hat{\nu}[m', m]^{1-p} + \hat{\nu}[n, n']^{1-p})^{-1/p}.$$

Let $m' \downarrow -M$, $n' \uparrow N$ and then make supremum with respect to $m = x \leq y = n$. We get the required assertion. \square

On the comparison of B_* and B^* , it is obvious that $B^* = B_*$ if $p = q$. In general, we have the following result.

Lemma 1.5 Let $q \geq p$. Then we have $B_* \leq B^* \leq 2^{1/p-1/q} B_*$.

Proof. Simply apply the c_r -inequality:

$$(\alpha + \beta)^r \leq 2^{(r-1) \vee 0} (\alpha^r + \beta^r).$$

(a) Set

$$\alpha = \hat{\nu}[-M, x]^{\frac{q(1-p)}{p}}, \quad \beta = \hat{\nu}[y, N]^{\frac{q(1-p)}{p}}, \quad r = \frac{p}{q} \in (0, 1].$$

It follows that

$$(\alpha + \beta)^{1/q} \leq (\alpha^{p/q} + \beta^{p/q})^{1/p},$$

and then $B^* \geq B_*$.

(b) Set

$$\alpha = \hat{\nu}[-M, x]^{1-p}, \quad \beta = \hat{\nu}[y, N]^{1-p}, \quad r = \frac{q}{p} \geq 1.$$

We have

$$(\alpha + \beta)^{1/p} \leq 2^{1/p-1/q} (\alpha^{q/p} + \beta^{q/p})^{1/p},$$

and then $B_* \geq 2^{1/q-1/p} B^*$. Certainly, in this case the assertion can also be deduced by Jensen's inequality. \square

We mention that even though their supremums are equivalent but in the proofs of Lemmas 1.3 and 1.4, the expressions of B^* and B_* are not exchangeable, because B_* does not own the homogeneous of that of B^* .

We are now ready to state our first main result.

Theorem 1.6 The optimal constant A in the Hardy-type inequality (1) satisfies

- (1) $A \leq k_{q,p} B^*$ for $1 < p \leq q < \infty$ once $\mu_{pp} = 0$, where $k_{q,p}$ is defined by (5), and
 (2) $A \geq B_*$ for $1 < p, q < \infty$, where

$$B^* = \sup_{-M \leq x \leq y \leq N} \left\{ \mu[x, y]^{1/q} \left(\hat{\nu}[-M, x]^{q(1-p)/p} + \hat{\nu}[y, N]^{q(1-p)/p} \right)^{-1/q} \right\},$$

$$B_* = \sup_{-M \leq x \leq y \leq N} \left\{ \mu[x, y]^{1/q} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-1/p} \right\}.$$

Moreover, we have $B_* \leq B^* \leq 2^{1/p-1/q} B_*$ when $q \geq p$.

Proof. The conclusions are combination of Lemmas 1.3–1.5. \square

It is interesting to have a look at the factor $k_{q,p}$ in Theorem 1.6. When $p = q$, the factor becomes

$$\left(\frac{q}{q-1} \right)^{(q-1)/q} q^{1/q}.$$

On $(1, \infty)$, it is unimodal having maximum 2 at $q = 2$ and decreases to 1 as $q \rightarrow 1$ or ∞ . More generally, the rough ratio of the upper and lower bounds is no more than

$$\left(1 + \frac{q}{p'} \right)^{1/q} \left(1 + \frac{p'}{q} \right)^{1/p'} 2^{1/p-1/q}$$

which is again ≤ 2 (for every $q \geq p$), having equality sign iff $p = q = 2$.

The study on the inequality (1) was began by P. Gurka in an unpublished paper using a common constant

$$\begin{aligned} \tilde{B} &= \sup_{-M \leq x \leq y \leq N} \left\{ \mu[x, y]^{\frac{1}{q}} \left(\hat{\nu}[-M, x]^{1-p} \vee \hat{\nu}[y, N]^{1-p} \right)^{-\frac{1}{p}} \right\} \\ &= \sup_{-M \leq x \leq y \leq N} \left\{ \mu[x, y]^{\frac{1}{q}} \left(\hat{\nu}[-M, x]^{\frac{p-1}{p}} \wedge \hat{\nu}[y, N]^{\frac{p-1}{p}} \right) \right\} \end{aligned}$$

and having a universal factor 8. This \tilde{B} is closely related to B_* : replacing “+” with “ \vee ”, we obtain \tilde{B} from B_* . Gurka’s result was then improved in [18; Theorem 8.2] with a smaller factor (unexplicit one ≈ 4.71 and explicit one $= 2\sqrt{6}$ in the case of $p = q = 2$). Note that using the inequalities

$$\alpha \vee \beta \leq \alpha + \beta \leq 2(\alpha \vee \beta),$$

from Theorem 1.6, it follows that we have lower and upper bounds replacing B^* and B_* by the same \tilde{B} with an additional factor $2^{-1/p}$ for the lower estimate. Then the factor becomes $2\sqrt{2}$ in the case of $p = q = 2$. Replacing $\alpha \vee \beta$ with $\alpha + \beta$ is an essential difference of the present paper from the previous ones

in the bilateral situation. Besides, the inequality (1) was also proved in [18; Theorem 8.8] with a common constant

$$\tilde{B} = \inf_{-M \leq \theta \leq N} \left\{ \left[\sup_{-M \leq x \leq \theta} \mu[x, \theta]^{\frac{1}{q}} \hat{\nu}[-M, x]^{\frac{p-1}{p}} \right] \vee \left[\sup_{\theta \leq y \leq N} \mu[\theta, y]^{\frac{1}{q}} \hat{\nu}[y, N]^{\frac{p-1}{p}} \right] \right\}$$

having a factor

$$2^{\frac{1}{p}} \left(1 + \frac{q}{p'} \right)^{\frac{1}{q}} \left(1 + \frac{p'}{q} \right)^{\frac{1}{p'}}$$

which has an additional factor is $2^{1/p}$ than (5). The last result is related to our splitting technique. All of these results use the assumption that $\mu \ll dx$ and $\nu \ll dx$.

Before moving further, we want to describe B_* and B_* more carefully. It also leads some quantities which are easier in practical computations. For this, we need some preparation. Assume that (7) holds. For each $x \in (-M, N)$, let $y(x)$ be the unique solution of the equation

$$\hat{\nu}[-M, x] = \hat{\nu}[y, N].$$

Next, let $m(\hat{\nu})$ be a solution to the equation

$$y(x) = x, \quad x \in (-M, N).$$

Thus, $m(\hat{\nu})$ is actually the median of the measure $\hat{\nu}$ (but not ν):

$$\hat{\nu}[-M, m] = \hat{\nu}[m, N].$$

Set

$$H_{\mu, \nu}(x, y) = \mu[x, y]^{1/q} \left[\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right]^{-1/p}, \\ -M \leq x \leq y \leq N.$$

Define

$$H^o = 2^{-1/p} \sup_{x \in (-M, m(\hat{\nu}))} \mu[x, y(x)]^{1/q} \hat{\nu}[-M, x]^{(p-1)/p}. \quad (10)$$

Denote by Γ be the limiting points of $H_{\mu, \nu}(x, y)$ as $\mu[y, N] = \infty$ or $\mu[-M, x] = \infty$, as well as the iterated limits if $\mu[-M, N] = \infty$ when $M = \infty = N$. Set

$$H^\partial = \begin{cases} \sup\{\gamma : \gamma \in \Gamma\} & \text{if } \Gamma \neq \emptyset \\ 0 & \text{if } \Gamma = \emptyset. \end{cases} \quad (11)$$

Clearly, $H^\partial = 0$ if $M, N < \infty$.

We are now ready to describe B^* and B_* in terms of H^o and H^∂ .

Lemma 1.7 Let (7) hold. Then we have

$$H^o \vee H^\partial \leq B_* \leq (2^{1/p} H^o) \vee H^\partial.$$

Proof. Rewrite H as

$$H_{\mu,\nu}(x, y) = \left[\frac{\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p}}{\mu[x, y]^{p/q}} \right]^{-1/p}.$$

under (7), because for finite x and y with $x \leq y$, we have

$$\frac{\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p}}{\mu[x, y]^{p/q}} \geq \frac{2}{\mu[x, y]^{p/q}} \left[\hat{\nu}[-M, x] \hat{\nu}[y, N] \right]^{(1-p)/2},$$

and the equality sign holds iff $\hat{\nu}[-M, x] = \hat{\nu}[y, N]$ which gives us the solution $y(x)$. Thus, we obtain

$$\inf_{x \leq y} \frac{\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p}}{\mu[x, y]^{p/q}} \leq 2 \inf_{x \leq m(\hat{\nu})} \frac{\hat{\nu}[-M, x]^{1-p}}{\mu[x, y(x)]^{p/q}}, \quad (12)$$

since $\{(x, y(x)) : x \leq m(\hat{\nu})\} \subset \{(x, y) : x, y \in (-M, N)\}$. This gives us a lower bound of the supremum of $H_{\mu,\nu}(x, y)$ over the set $\{(x, y) : x < y, \mu[x, y] < \infty\}$. Next, we have

$$\begin{aligned} \inf_{x \leq y} \frac{\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p}}{\mu[x, y]^{p/q}} &= \inf_{x \leq y} \left\{ \frac{\hat{\nu}[-M, x]^{1-p}}{\mu[x, y]^{p/q}} + \frac{\hat{\nu}[y, N]^{1-p}}{\mu[x, y]^{p/q}} \right\} \\ &\geq \inf_{x \leq y} \left\{ \frac{\hat{\nu}[-M, x]^{1-p}}{\mu[x, y]^{p/q}} \vee \frac{\hat{\nu}[y, N]^{1-p}}{\mu[x, y]^{p/q}} \right\} \\ &=: \xi. \end{aligned}$$

Without loss of generality, assume that $M, N < \infty$. Because of the continuity of the involved functions, the minimum ξ can be achieved at some pair (x_0, y_0) . We now prove that (x_0, y_0) should be located at the surface where the two terms in the last $\{\dots\}$ are equal. Otherwise, without loss of generality, assume that

$$\varepsilon := \mu[x_0, y_0]^{-p/q} \hat{\nu}[-M, x_0]^{1-p} - \mu[x_0, y_0]^{-p/q} \hat{\nu}[y_0, N]^{1-p} > 0.$$

Let $\bar{y} > y_0$ be sufficiently close to y_0 . Then we have

$$\mu[x_0, \bar{y}]^{-p/q} \hat{\nu}[-M, x_0]^{1-p} < \mu[x_0, y_0]^{-p/q} \hat{\nu}[-M, x_0]^{1-p}$$

(here we have used the preassumption that μ is positive on each subinterval) and

$$\mu[x_0, \bar{y}]^{-p/q} \hat{\nu}[\bar{y}, N]^{1-p} < \mu[x_0, y_0]^{-p/q} \hat{\nu}[y_0, N]^{1-p} + \varepsilon/2,$$

due to the continuity of the involved functions. We have thus obtained a pair (x_0, \bar{y}) with $x_0 < \bar{y}$ such that

$$\left\{ \left[\mu[x_0, \bar{y}]^{-p/q} \hat{\nu}[-M, x_0]^{1-p} \right] \vee \left[\mu[x_0, \bar{y}]^{-p/q} \hat{\nu}[\bar{y}, N]^{1-p} \right] \right\} < \xi.$$

This is a contradiction to the minimum property of ξ . Therefore, we obtain

$$\inf_{x \leq y} \mu[x, y]^{-p/q} \left[\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right] \geq \inf_{x \leq m(\hat{\nu})} \frac{\hat{\nu}[-M, x]^{1-p}}{\mu[x, y(x)]^{p/q}}. \quad (13)$$

From this, we obtain an upper bound of the supremum of $H_{\mu, \nu}(x, y)$ over the set $\{(x, y) : x < y, \mu[x, y] < \infty\}$ in terms of H^o up to a factor $2^{-1/p}$. In other words, we have worked out the case that the supremum is achieved inside of the interval. In general, it may be achieved at the ∞ -boundaries (at which $\mu[y, N] = \infty$ or $\mu[-M, x] = \infty$). This leads to the boundary condition H^∂ , when one of M and N is infinite. Combining these two parts together, we get the estimates of B_* under (7). \square

An easier way to understand what was going on in the last proof is to look at the following simple example. Consider functions $f(x) = 2x$ and $g(x) = 3 - x$ on $[0, 2]$. They intersect uniquely at the point $x^* = 1$. Then we have

$$2\sqrt{f(x^*)g(x^*)} = 4 > \inf_{x \in [0, 2]} (f(x) + g(x)) = 3 > \inf_{x \in [0, 2]} [f(x) \vee g(x)] = f(x^*) = 2.$$

If we rewrite the first term as $2 \inf_{x \in [0, 2]} (f(x) \vee g(x))$, then it becomes obvious that the middle term can be bounded by the first and the last ones. Certainly, the bounds are usually not sharp.

Lemma 1.8 Let (7) hold. Then we have

$$(2^{1/p-1/q} H^o) \vee H^\partial \leq B^* \leq (2^{1/p} H^o) \vee H^\partial.$$

Proof. Note that the difference of B^* and B_* is only the summation terms. If one of the terms in the sum is ignored, then the remaining terms coincide with each other. Thus, the boundary condition H^∂ is the same for B^* and B_* . When $M, N < \infty$, the proof of the comparison of B^* with H^o is similar to the last one. \square

We remark that in the degenerated case that (7) does not hold, say $\hat{\nu}[y, N] = \infty$, then we have obviously that $B_* = B^*$.

As a combination of the last two lemmas, we obtain the following simple criterion.

Corollary 1.9 The Hardy-type inequality (1) holds iff $H^o \vee H^\partial < \infty$.

Proof. When (7) holds, the assertion follows from the last two lemmas. Note that if $\hat{\nu}[y, N] = \infty$ for instance, we have $B^* = B_*$ and so the assertion is described by H^∂ only. \square

We now extend Theorem 1.6 to a more general setup which is mainly used in interpolation of L^p -spaces. For this, we need a class of normed linear spaces

$(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$ consisting of real Borel measurable functions on a measurable space (X, \mathcal{X}, μ) . We now modify the hypotheses on the normed linear spaces given in [5; Chapter 7] as follows.

Hypotheses 1.10 (H1) In the case that $\mu(X) = \infty$, $\mathbb{1}_K \in \mathbb{B}$ for all compact K . Otherwise, $1 \in \mathbb{B}$.

(H2) If $h \in \mathbb{B}$ and $|f| \leq h$, then $f \in \mathbb{B}$.

(H3) $\|f\|_{\mathbb{B}} = \sup_{g \in \mathcal{G}} \int_X |f|g d\mu$,

where \mathcal{G} , to be specified case by case, is a class of nonnegative \mathcal{X} -measurable functions. A typical example is $\mathcal{G} = \{\mathbb{1}\}$ and then $\mathbb{B} = L^1(\mu)$. In what follows, the measure space (X, \mathcal{X}, μ) is fixed to be $([-M, N], \mathcal{B}([-M, N]), \mu)$. We often use the dual representation of the norm

$$\left(\int_{-M}^N |f|^r d\mu \right)^{1/r} = \sup_{g \in \text{The unit ball in } L^{r'}(\mu)} \int_{-M}^N |f|g d\mu,$$

where r' is the conjugate number of r (≥ 1). Throughout this paper, we assume (H1)–(H3) for $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$ without mentioned again.

For simplicity, we write the L^p -norm with respect to μ as $\|\cdot\|_{\mu, p}$. If necessary, we also write $\|\cdot\|_{\alpha, \beta; \mu, p}$ to indicate the interval $[\alpha, \beta]$.

Theorem 1.11 Let \mathcal{G} satisfy Hypotheses 1.10 and consider the Hardy-type inequality

$$\|f^q\|_{\mathbb{B}}^{1/q} \leq A_{\mathbb{B}} \|f'\|_{\nu, p}, \quad f(-M) = 0 \text{ and } f(N) = 0.$$

(1) Then the optimal constant $A_{\mathbb{B}}$ satisfies

$$A_{\mathbb{B}} \leq k_{q,p} B_{\mathbb{B}}^* \quad \text{for } 1 < p \leq q < \infty \text{ once } \mu_{\text{pp}} = 0, \text{ and}$$

(2) $A_{\mathbb{B}} \geq B_{\mathbb{B}*}$ for $1 < p, q < \infty$, where

$$B_{\mathbb{B}}^* = \sup_{-M \leq x \leq y \leq N} \left\{ \|\mathbb{1}_{[x,y]}\|_{\mathbb{B}}^{\frac{1}{q}} \left(\hat{\nu}[-M, x]^{\frac{q(1-p)}{p}} + \hat{\nu}[y, N]^{\frac{q(1-p)}{p}} \right)^{-\frac{1}{q}} \right\},$$

$$B_{\mathbb{B}*} = \sup_{-M \leq x \leq y \leq N} \left\{ \|\mathbb{1}_{[x,y]}\|_{\mathbb{B}}^{\frac{1}{q}} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-\frac{1}{p}} \right\}.$$

Moreover, we have $B_{\mathbb{B}*} \leq B_{\mathbb{B}}^* \leq 2^{1/p-1/q} B_{\mathbb{B}*}$ whenever $q \geq p$.

Proof. Let $g \in \mathcal{G}$. Without loss of generality, assume that $g > 0$. For the pair $\mu_g := g\mu$ and ν , by Theorem 1.6, we know that the corresponding optimal constant A_g in (1) satisfies

$$B_{g*} \leq A_g \leq k_{q,p} B_g^*,$$

where

$$B_g^* = \sup_{-M \leq x \leq y \leq N} \left\{ \mu_g[x, y]^{\frac{1}{q}} \left(\hat{\nu}[-M, x]^{\frac{q(1-p)}{p}} + \hat{\nu}[y, N]^{\frac{q(1-p)}{p}} \right)^{-\frac{1}{q}} \right\},$$

$$B_{g*} = \sup_{-M \leq x \leq y \leq N} \left\{ \mu_g[x, y]^{\frac{1}{q}} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-\frac{1}{p}} \right\}.$$

Hence

$$\begin{aligned} \sup_{g \in \mathcal{G}} B_{g*} &= \sup_{x \leq y} \sup_g \left\{ \mu_g[x, y]^{1/q} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-1/p} \right\} \\ &= \sup_{x \leq y} \left\{ \left(\sup_g \mu_g[x, y] \right)^{1/q} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-1/p} \right\} \\ &= \sup_{x \leq y} \left\{ \|\mathbb{1}_{[x, y]}\|_{\mathbb{B}}^{1/q} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-1/p} \right\} \\ &= B_{\mathbb{B}*}. \end{aligned}$$

Similarly, we have $\sup_{g \in \mathcal{G}} B_g^* = B_{\mathbb{B}}^*$. From these facts, we obtain the estimates of $A_{\mathbb{B}} = \sup_{g \in \mathcal{G}} A_g$ immediately. The last assertion then follows from Lemma 1.5 (or its proof). \square

As an application of Theorem 1.11, it follows that the optimal constant $A_{\mathbb{B}}$ in the inequality

$$\|f^p\|_{\mathbb{B}}^{1/p} \leq A_{\mathbb{B}} \|f'\|_{\nu, p}, \quad f(-M) = 0 \text{ and } f(N) = 0$$

satisfies

$$B_{\mathbb{B}*} \leq A_{\mathbb{B}} \leq k_{p,p} B_{\mathbb{B}*}, \quad (14)$$

where

$$B_{\mathbb{B}*} = \sup_{-M \leq x \leq y \leq N} \left\{ \|\mathbb{1}_{[x, y]}\|_{\mathbb{B}}^{\frac{1}{p}} \left(\hat{\nu}[-M, x]^{1-p} + \hat{\nu}[y, N]^{1-p} \right)^{-\frac{1}{p}} \right\}.$$

Clearly, this result is simpler than Theorem 1.11. Now, applying this result to $\mathbb{B} = L^{q/(q-p)}$ ($q > p$, where $q/(q-p)$ is the conjugate number of q/p), we return to Theorem 1.6 with a factor $k_{p,p}$ different from $k_{q,p}$. In other words, we have arrived at a conclusion that the optimal constant A in the Hardy-type inequality (1) (with $q \geq p$) satisfies

$$B_* \leq A \leq k_{p,p} B_*, \quad (15)$$

where B_* is the same as in Theorem 1.6. However, as we will see from Example 1.13 below that this result (15) may be less sharp than Theorem 1.6. Thus, lifting the left-hand side of the Hardy-type inequality from $L^p(\mu)$ to \mathbb{B} does keep the constant $k_{p,p}$ but can not improve it. We have thus explained the reason why on the left-hand side of the first inequality in Theorem 1.11, we

use q but not p . In the discrete context and $p = 2$, the conclusion (14) was presented by [6; Theorem 8.2].

In view of the direct proof for the lower estimate of A , the restriction $\mu_{pp} = 0$ coming from our splitting technique seems unnecessary. As just mentioned, the conclusion (14) was proved in the discrete situation when $p = 2$ by [6; Theorem 8.2]. Its proof should be meaningful in the present continuous case (cf. [5; Proof of Corollary 7.6], simply replacing the function $e^{C(x)}$ there by the one $h(x)$ here). Hence for (14) in the case of $p = 2$, the condition $\mu_{pp} = 0$ is again not needed. Thus, one may remove this condition by the capacitary method first for $q = p$, if possible, and then extend to general normed linear space \mathbb{B} as shown above. The present proof depends heavily on the splitting property as shown by the first step of the proof of Lemma 1.3 and the comments below Lemma 1.3. This is one of the reasons why the case of $q < p$ is missed here. Actually, the last case is a rather different story, refer to [18; Theorem 8.17] or [16; pages 50-51].

To illustrate the application of Theorem 1.6, we study two examples.

Example 1.12 Let $(-M, N) = (0, 1)$ and $d\mu = d\nu = dx$. Then $H^\partial = 0$ and

$$B_* = H^o = \frac{1}{2} \left(\frac{p}{p-q+pq} \right)^{\frac{1}{q}} \left(\frac{(p-1)q}{p-q+pq} \right)^{\frac{p-1}{p}},$$

$$B^* = 2^{1/p-1/q} H^o, \quad 1 < p \leq q < \infty.$$

When $p = q = 2$, it is known that $A = \pi^{-1}$ and $B^* = B_* = 1/4$ (cf. [7; Example 5.2]).

Proof. We have $h \equiv 1$, $y(x) = 1 - x$, and $m(\hat{\nu}) = 1/2$. The function

$$H_{\mu,\nu}(x, y(x)) = 2^{-\frac{1}{p}} \mu[x, y(x)]^{\frac{1}{q}} \hat{\nu}[0, x]^{\frac{p-1}{p}} = 2^{-\frac{1}{p}} (1-2x)^{\frac{1}{q}} x^{\frac{p-1}{p}}$$

achieves its maximum at

$$x = \frac{1}{2} \left(1 + \frac{p}{(p-1)q} \right)^{-1} < \frac{1}{2}.$$

From this, we obtain H^o . Next, note that

$$B_* = \sup_{x \leq y} \frac{(1 - (1-y) - x)^{1/q}}{(x^{1-p} + (1-y)^{1-p})^{1/p}},$$

$$B^* = \sup_{x \leq y} \frac{(1 - (1-y) - x)^{1/q}}{(x^{q(1-p)/p} + (1-y)^{q(1-p)/p})^{1/q}}.$$

Both of them are symmetric in x and $1-y$. Hence $B_* = H^o$ and furthermore $B^* = 2^{1/p-1/q} H^o$. \square

The next example illustrates the role played by H^∂ .

Example 1.13 Let $\mu(dx) = dx$ and $\nu(dx) = x^2 dx$ on $(1, \infty)$. Assume that $p \in (1, 3)$. Then the inequality does not hold if $q \in [p, p/(3-p))$. Otherwise, the inequality holds with

$$B_* = B^* = H^\partial = \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} \quad \text{if } q = \frac{p}{3-p} \text{ and } p \in [2, 3);$$

B_* and B^* are bounded in terms of H^o if $q > \frac{p}{3-p}$ and $q \geq p$,

where

$$H^o = 2^{-\frac{1}{p}} \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} \sup_{x \in (1, 2^{(3-p)/(p-1)}]} \left[\left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p-3}} - x \right]^{\frac{1}{q}} \left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p}}.$$

When $p = q = 2$, we have $B^* = B_* = 1$. In this case, $A = 2$ and so the upper estimate $2B^*$ in Theorem 1.6 is exact (cf. [7; Example 5.4]). For fixed $p = 2$, when q varies from 2.01 to 4.8, the five quantities we have worked so far are shown in Figure 1. The ratio of the upper and lower bounds is decreasing in q but no more than 2. Note that we have a common lower bound B_* in Theorem 1.6 and (15), but the upper bound in Theorem 1.6 is better than that in (15).

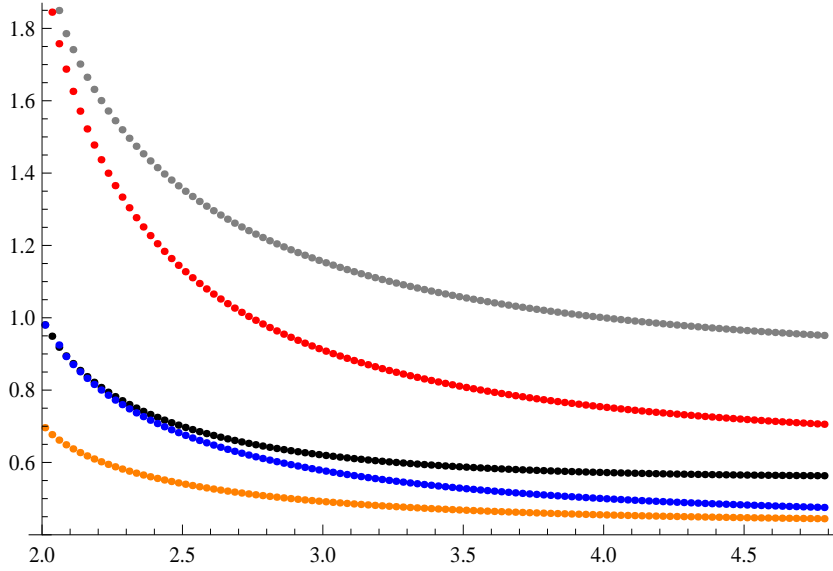


Figure 1 The curves from bottom to top are H^o , B_* , B^* , $k_{q,p}B^*$, and $k_{p,p}B_*$ respectively.

Proof. We have $h(x) = x^{-2/(p-1)}$. Then

$$\int_1^x h = \frac{p-1}{3-p} \left(1 - x^{\frac{p-3}{p-1}}\right), \quad \int_y^\infty h = \frac{p-1}{3-p} y^{\frac{p-3}{p-1}},$$

where and in what follows, the Lebesgue measure dz is omitted. Hence

$$y(x) = \left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p-3}}, \quad m(\hat{\nu}) = 2^{\frac{3-p}{p-1}},$$

$$H_{\mu,\nu}(x, y) = \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} (y-x)^{\frac{1}{q}} \left[\left(1 - x^{\frac{p-3}{p-1}}\right)^{1-p} + y^{3-p} \right]^{-\frac{1}{p}}.$$

In particular,

$$\overline{\lim}_{y \rightarrow \infty} H_{\mu,\nu}(x, y) = \lim_{y \rightarrow \infty} \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} y^{1+\frac{1}{q}-\frac{3}{p}} = \begin{cases} \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} & \text{if } 1 + \frac{1}{q} = \frac{3}{p} \\ \infty & \text{if } 1 + \frac{1}{q} > \frac{3}{p} \\ 0 & \text{if } 1 + \frac{1}{q} < \frac{3}{p}. \end{cases}$$

The right-hand side is our H^∂ . Thus, if $1 + \frac{1}{q} > \frac{3}{p}$, then $B_* = \infty$. Next, we have

$$H_{\mu,\nu}(x, y(x)) = 2^{-\frac{1}{p}} \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} \left[\left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p-3}} - x \right]^{\frac{1}{q}} \left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p}}.$$

Then

$$H^o = 2^{-\frac{1}{p}} \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} \sup_{x \in (1, 2^{(3-p)/(p-1)})} \left[\left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p-3}} - x \right]^{\frac{1}{q}} \left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{p-1}{p}}.$$

The point here is that $H^o \leq H^\partial$ and $2^{1/p} H^o \leq H^\partial$ in the case of $q = p/(3-p)$ and $p \geq 2$. Besides, we have

$$B_* = \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} \sup_{x < y} \left\{ (y-x)^{\frac{1}{q}} \left[\left(1 - x^{\frac{p-3}{p-1}}\right)^{1-p} + y^{3-p} \right]^{-\frac{1}{p}} \right\},$$

$$B^* = \left(\frac{p-1}{3-p}\right)^{\frac{p-1}{p}} \sup_{x < y} \left\{ (y-x)^{\frac{1}{q}} \left[\left(1 - x^{\frac{p-3}{p-1}}\right)^{\frac{q(1-p)}{p}} + y^{\frac{q(3-p)}{p}} \right]^{-\frac{1}{q}} \right\}.$$

Finally, numerical computation gives us the quantities B^* et al, as shown in Figure 1. \square

2 The case of mean zero.

Throughout this section, we assume that $\mu[-M, N] < \infty$ and define a probability measure $\pi = (\mu[-M, N])^{-1} \mu$. In probabilistic language, we are going to study the ergodic case of the corresponding processes. Corresponding to the three inequalities given at the beginning of the last section, we now study the inequality

$$\left(\int_{-M}^N |f - \pi(f)|^q d\mu \right)^{1/q} \leq A \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p}, \quad (16)$$

where $\pi(f) = \int f d\pi$, in terms of

$$\begin{aligned} \left(\int_{\theta}^N |f|^q d\mu \right)^{1/q} &\leq A_{\theta}^{+} \left(\int_{\theta}^N |f'|^p d\nu \right)^{1/p}, & f(\theta) = 0, \\ \left(\int_{-M}^{\theta} |f|^q d\mu \right)^{1/q} &\leq A_{\theta}^{-} \left(\int_{-M}^{\theta} |f'|^p d\nu \right)^{1/p}, & f(\theta) = 0. \end{aligned}$$

To save our notation, without any confusion, we use the same notation A , A_{θ}^{\pm} and so on as in the last section.

Before moving further, let us mention the spectral meaning of (1) and (16). Suppose that $\mu \ll dx$ and $\nu \ll dx$, denote by $u = d\mu/dx$ and $v = d\nu/dx$. Then the inverse of the optimal constant A in (1) and (16), when $q = p$, corresponds to the infimum $\lambda^{1/p}$ of the nontrivial spectrum of

$$(v|f'|^{p-1} \operatorname{sgn}(f'))' = -\lambda u|f|^{q-1} \operatorname{sgn}(f)$$

with boundary condition $f(-M) = 0 = f(N)$ and $f'(-M) = 0 = f'(N)$ (when $M, N < \infty$), respectively. The word “bilateral” in the title means that a same boundary condition is endowed at two endpoints of the interval. The spectral point of view has played a crucial role in our previous study. For instance, it appears in each of the papers [3] – [9].

To study (16), we start again at the splitting technique. We begin with the easier case: the lower estimate. It is indeed easier than the one studied in the last section.

Lemma 2.1 Let $1 \leq p \leq q < \infty$. Then we have

$$A \geq 2^{1/q-1/p} \sup_{\theta \in [-M, N]} (A_{\theta}^{-} \wedge A_{\theta}^{+}).$$

Proof. Fix $\theta \in [-M, N]$. Let f_{-} satisfy $f_{-}|_{[\theta, N]} = 0$,

$$\int_{-M}^{\theta} |f_{-}|^q d\mu = 1, \quad \text{and} \quad \left(\int_{-M}^{\theta} |f'_{-}|^p d\nu \right)^{1/p} < (A_{\theta}^{-})^{-1} + \varepsilon.$$

Let f_{+} satisfy $f_{+}|_{[-M, \theta]} = 0$,

$$\int_{\theta}^N |f_{+}|^q d\mu = 1, \quad \text{and} \quad \left(\int_{\theta}^N |f'_{+}|^p d\nu \right)^{1/p} < (A_{\theta}^{+})^{-1} + \varepsilon.$$

Set $f = cf_{-} + f_{+}$, where $c = -\pi(f_{+})/\pi(f_{-})$. Then $\pi(f) = 0$,

$$1 + |c|^q = \int_{-M}^{\theta} |cf_{-}|^q d\mu + \int_{\theta}^N |f_{+}|^q d\mu = \int_{-M}^N |f|^q d\mu,$$

and

$$\begin{aligned} \int_{-M}^N |f'|^p d\nu &= |c|^p \int_{-M}^{\theta} |f'_-|^p d\nu + \int_{\theta}^N |f'_+|^p d\nu \\ &\leq |c|^p \left((A_{\theta}^-)^{-1} + \varepsilon \right)^p + \left((A_{\theta}^+)^{-1} + \varepsilon \right)^p \\ &\leq \left((A_{\theta}^-)^{-1} \vee (A_{\theta}^+)^{-1} + \varepsilon \right)^p (1 + |c|^p). \end{aligned}$$

Hence

$$\begin{aligned} \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p} &\leq \left((A_{\theta}^-)^{-1} \vee (A_{\theta}^+)^{-1} + \varepsilon \right) (1 + |c|^p)^{1/p} \\ &\leq 2^{1/p-1/q} \left((A_{\theta}^-)^{-1} \vee (A_{\theta}^+)^{-1} + \varepsilon \right) (1 + |c|^q)^{1/q} \\ &\quad \text{(by Jensen's inequality requiring } q \geq p) \\ &= 2^{1/p-1/q} \left((A_{\theta}^-)^{-1} \vee (A_{\theta}^+)^{-1} + \varepsilon \right) \left(\int_{-M}^N |f|^q d\mu \right)^{1/q}. \end{aligned}$$

Thus

$$A \geq 2^{1/q-1/p} (A_{\theta}^- \wedge A_{\theta}^+).$$

Since θ is arbitrary, we obtain the lower bound of A . \square

The upper bound of A is harder than the lower one just studied. But the first step is still easy. Given f and $\theta \in (-M, N)$, let $\tilde{f} = f - f(\theta)$. Then

$$\begin{aligned} \int_{-M}^N |f'|^p d\nu &= \int_{-M}^{\theta} |\tilde{f}'|^p d\nu + \int_{\theta}^N |\tilde{f}'|^p d\nu \\ &\geq (A_{\theta}^-)^{-p} \left(\int_{-M}^{\theta} |\tilde{f}|^q d\mu \right)^{p/q} + (A_{\theta}^+)^{-p} \left(\int_{\theta}^N |\tilde{f}|^q d\mu \right)^{p/q} \\ &\geq \left[(A_{\theta}^-)^{-p} \wedge (A_{\theta}^+)^{-p} \right] \left[\left(\int_{-M}^{\theta} |\tilde{f}|^q d\mu \right)^{p/q} + \left(\int_{\theta}^N |\tilde{f}|^q d\mu \right)^{p/q} \right] \\ &\geq (2^{(p/q-1) \vee 0})^{-1} \left[(A_{\theta}^-)^{-p} \wedge (A_{\theta}^+)^{-p} \right] \left(\int_{-M}^N |\tilde{f}|^q d\mu \right)^{p/q} \\ &\quad \text{(by } c_r\text{-inequality).} \end{aligned}$$

Our aim is to replace $|\tilde{f}|^q$ on the right-hand side with $|f - \pi(f)|^q$. This is true in the case of $q = 2$ since

$$\inf_{c \in \mathbb{R}} \int_{-M}^N (f - c)^2 d\mu = \int_{-M}^N (f - \pi(f))^2 d\mu.$$

Unfortunately, it does not work for general q . Anyhow, when $q = 2$, we have

$$\int_{-M}^N |f'|^p d\nu \geq (2^{(p/2-1) \vee 0})^{-1} \left[(A_{\theta}^-)^{-p} \wedge (A_{\theta}^+)^{-p} \right] \left(\int_{-M}^N |f - \pi(f)|^2 d\mu \right)^{p/2}.$$

Since θ is arbitrary, we obtain

$$\begin{aligned} (2^{(1/2-1/p)\vee 0}) \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p} &\geq \sup_{\theta \in (-M, N)} \left[(A_\theta^-)^{-1} \wedge (A_\theta^+)^{-1} \right] \\ &\quad \times \left(\int_{-M}^N |f - \pi(f)|^2 d\mu \right)^{1/2}. \end{aligned}$$

Next, since f is arbitrary, it follows that

$$A \leq 2^{(1/2-1/p)\vee 0} \inf_{\theta \in (-M, N)} (A_\theta^- \vee A_\theta^+).$$

Up to now, the proof is similar to [3; Theorems 3.3 and 3.4] in the specific case that $q = 2$. For general $q \geq 2$, we have luckily a different approach (cf. [5; Chapter 6] and references therein). Note that we have already proved that if the measure μ is replaced by $\mu_g := g\mu$ for a nonnegative function g on $[-M, N]$, then the optimal constant A_g in the inequality

$$\left(\int_{-M}^N |f - \pi(f)|^2 d\mu_g \right)^{1/2} \leq A_g \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p}$$

obeys

$$A_g \leq 2^{(1/2-1/p)\vee 0} \inf_{\theta \in (-M, N)} (A_g^{\theta, -} \vee A_g^{\theta, +}),$$

where $A_g^{\theta, \pm}$ is obtained from A_θ^\pm replacing μ with μ_g . From now on in this proof, the constants A_g , $A_g^{\theta, \pm}$, and $B_g^{\theta, \pm}$ are used for μ_g in the specified case that $q = 2$ only. Note that for $A_g^{\theta, +}$ for instance, the function g can be replaced by $g\mathbb{1}_{[\theta, N]}$.

Even though we are now mainly working on the L^q -case to which \mathcal{G} is the set of functions in the unit ball of $L^{\frac{q}{q-2}}(\mu)$ (where $\frac{q}{q-2}$ is the conjugate number of $q/2$):

$$\left(\int_{-M}^N |f - \pi(f)|^q d\mu \right)^{2/q} = \sup_{g \in \mathcal{G}} \int_{-M}^N |f - \pi(f)|^2 g d\mu = \sup_{g \in \mathcal{G}} \int_{-M}^N |f - \pi(f)|^2 d\mu_g,$$

at the moment, we allow \mathcal{G} to be general in the setup of Hypotheses 1.10:

$$\|(f - \pi(f))^2\|_{\mathbb{B}}^{1/2} \leq A_{\mathbb{B}} \|f'\|_{\nu, p}. \quad (17)$$

We have

$$\begin{aligned} A_{\mathbb{B}} &= \sup_{g \in \mathcal{G}} A_g \leq \sup_{g \in \mathcal{G}} \inf_{\theta \in (-M, N)} (A_g^{\theta, -} \vee A_g^{\theta, +}) \\ &\leq \inf_{\theta \in (-M, N)} \left[\left(\sup_{g \in \mathcal{G}_\theta^-} A_g^{\theta, -} \right) \vee \left(\sup_{g \in \mathcal{G}_\theta^+} A_g^{\theta, +} \right) \right], \end{aligned}$$

where

$$\mathcal{G}_\theta^- = \{g|_{[-M, \theta]} : g \in \mathcal{G}\}, \quad \mathcal{G}_\theta^+ = \{g|_{[\theta, N]} : g \in \mathcal{G}\}.$$

Here is a technical point. Because on the left-hand side of (17), we start at $q = 2$. This leads to the restriction that $p \in (1, 2]$ since we need $q \geq p$ in order to use the basic estimates in terms of B_θ^\pm given in the next proof. Anyhow, we have proved the first assertion of the next result.

Lemma 2.2 Let \mathcal{G} satisfy Hypotheses 1.10. Then for $p \in (1, 2]$, we have

$$A_\mathbb{B} \leq \inf_{\theta \in (-M, N)} \left[\left(\sup_{g \in \mathcal{G}_\theta^-} A_g^{\theta, -} \right) \vee \left(\sup_{g \in \mathcal{G}_\theta^+} A_g^{\theta, +} \right) \right].$$

Moreover,

$$\sup_{g \in \mathcal{G}_\theta^\pm} A_g^{\theta, \pm} \leq k_{2,p} B_\mathbb{B}^{\theta, \pm},$$

where

$$\begin{aligned} B_\mathbb{B}^{\theta, +} &= \sup_{r \in (\theta, N)} \|\mathbb{1}_{[r, N]}\|_\mathbb{B}^{1/2} \hat{\nu}[\theta, r]^{(p-1)/p}, \\ B_\mathbb{B}^{\theta, -} &= \sup_{r \in (-M, \theta)} \|\mathbb{1}_{[-M, r]}\|_\mathbb{B}^{1/2} \hat{\nu}[r, \theta]^{(p-1)/p}. \end{aligned}$$

Proof. By [18; Theorem 1.14] and [16; §1.3, Theorem 1] (see also [2; Theorem 8] and [13; Theorem 2] in which the factor $k_{q,p}$ may be different), we have for general $1 < p \leq q < \infty$ that

$$\begin{aligned} B_\theta^\pm &\leq A_\theta^\pm \leq k_{q,p} B_\theta^\pm, \quad 1 < p \leq q < \infty, \\ B_\theta^+ &= \sup_{r \in (\theta, N)} \mu[r, N]^{1/q} \hat{\nu}[\theta, r]^{(p-1)/p}, \\ B_\theta^- &= \sup_{r \in (-M, \theta)} \mu[-M, r]^{1/q} \hat{\nu}[r, \theta]^{(p-1)/p}, \end{aligned}$$

where $\hat{\nu}$ is the same as in the last section. It remains to estimate $\sup_{g \in \mathcal{G}_\theta^+} A_g^{\theta, +}$ for instance. First, we have for $q = 2$ that

$$\sup_{g \in \mathcal{G}_\theta^+} A_g^{\theta, +} \leq k_{2,p} \sup_{g \in \mathcal{G}_\theta^+} B_g^{\theta, +}.$$

Next, we have

$$\begin{aligned} \sup_{g \in \mathcal{G}_\theta^+} B_g^{\theta, +} &= \sup_{g \in \mathcal{G}_\theta^+} \sup_{r \in (\theta, N)} \mu_g[r, N]^{1/2} \hat{\nu}[\theta, r]^{(p-1)/p} \\ &= \sup_{r \in (\theta, N)} \left(\sup_{g \in \mathcal{G}_\theta^+} \mu_g[r, N] \right)^{1/2} \hat{\nu}[\theta, r]^{(p-1)/p} \\ &= B_\mathbb{B}^{\theta, +}. \end{aligned}$$

Similar computation holds for $\sup_{g \in \mathcal{G}_\theta^-} B_g^{\theta, -}$. Combining these facts with the first assertion gives us the second one of the lemma. \square

As mentioned in the last section, without loss of generality, we can assume that μ is positive on each subinterval.

Here is our upper estimate.

Lemma 2.3 Let $\mu_{\text{pp}} = 0$ and \mathcal{G} satisfy Hypotheses 1.10. Then for $p \in (1, 2]$, we have

$$A_{\mathbb{B}} \leq k_{2,p} B_{\mathbb{B}}^*,$$

where the constant $B_{\mathbb{B}}^*$ is defined by

$$B_{\mathbb{B}}^{*- \frac{p}{p-1}} = \inf_{x < y} \left[\|\mathbb{1}_{[-M, x]}\|_{\mathbb{B}}^{-\frac{p}{2(p-1)}} + \|\mathbb{1}_{[y, N]}\|_{\mathbb{B}}^{-\frac{p}{2(p-1)}} \right] \hat{\nu}[x, y]^{-1}.$$

Proof. Write

$$B_{\mathbb{B}}^{*- \frac{p}{p-1}} = \sup_{x < y} \frac{\hat{\nu}[x, \theta] + \hat{\nu}(\theta, y)}{\|\mathbb{1}_{[-M, x]}\|_{\mathbb{B}}^{-\frac{p}{2(p-1)}} + \|\mathbb{1}_{[y, N]}\|_{\mathbb{B}}^{-\frac{p}{2(p-1)}}}.$$

Similar to the proof of Lemma 1.3, the assertion follows by using Lemmas 1.2 and 2.2. Here we may need the approximating procedure by finite M and N . \square

The following result is on the lower estimate of A . Its proof is new even in the special case that $p = q = 2$. Note that $\mu[x, y] = \mu(x, y)$ whenever $\mu_{\text{pp}} = 0$.

Lemma 2.4 Let $\mu_{\text{pp}} = 0$. Then for $1 < p, q < \infty$, the optimal constant A in (16) satisfies

$$\begin{aligned} A &\geq \sup_{-M \leq x < y \leq N} \left\{ \left[\mu[-M, x]^{\frac{1}{1-q}} + \mu[y, N]^{\frac{1}{1-q}} \right]^{\frac{1-q}{q}} \hat{\nu}[x, y]^{\frac{p-1}{p}} \right\} \\ &=: B_*. \end{aligned} \tag{18}$$

Proof. Given $m, n \in (-M, N)$ with $m < n$, let $\bar{\theta} = \bar{\theta}(m, n)$ be the unique solution to the equation

$$\mu[-M, m] \hat{\nu}[m, \theta] + \int_m^\theta \mu(dx) \hat{\nu}[x, \theta] = \mu[n, N] \hat{\nu}[\theta, n] + \int_\theta^n \mu(dx) \hat{\nu}[\theta, x], \quad \theta \in (m, n).$$

The existence of the solution is clear since μ is continuous, when θ varies from m to n , the left-hand side goes from 0 to a positive number and the right-hand side goes from a positive number to zero. Next, define

$$f(x) = -\mathbb{1}_{\{x \leq \bar{\theta}\}} \hat{\nu}[m \vee x, \bar{\theta}] + \mathbb{1}_{\{x > \bar{\theta}\}} \hat{\nu}[\bar{\theta}, n \wedge x], \quad x \in [-M, N].$$

Then $\mu(f) = 0$ by definition of $\bar{\theta}$. Clearly, f is absolutely continuous. On the one hand, we have

$$\begin{aligned} \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p} &= \left(\int_m^{\bar{\theta}} h^p d\nu + \int_{\bar{\theta}}^n h^p d\nu \right)^{1/p} \\ &= (\hat{\nu}[m, \bar{\theta}] + \hat{\nu}[\bar{\theta}, n])^{1/p} \\ &= \hat{\nu}[m, n]^{1/p}. \end{aligned} \quad (19)$$

Here in the second step, we have once again ignored the singular part of ν . On the other hand, we have

$$\begin{aligned} \int_{-M}^N |f - \pi(f)|^q d\mu &= \int_{-M}^N |f|^q d\mu \\ &> \int_{-M}^m |f|^q d\mu + \int_n^N |f|^q d\mu \\ &= \mu[-M, m] \hat{\nu}[m, \bar{\theta}]^q + \mu[n, N] \hat{\nu}[\bar{\theta}, n]^q. \end{aligned} \quad (20)$$

Now we have naturally, as in proof (a) of Proposition 1.1, that

$$\begin{aligned} \text{RHS of (20)} &\geq (\mu[-M, m] \wedge \mu[n, N]) (\hat{\nu}[m, \bar{\theta}]^q + \hat{\nu}[\bar{\theta}, n]^q) \\ &\geq 2^{1-q} (\mu[-M, m] \wedge \mu[n, N]) \hat{\nu}[m, n]^q. \end{aligned}$$

However, such a lower bound is quite rough for our purpose so we need a different approach. Note that the function

$$\gamma(x) = \alpha x^q + \beta(1-x)^q, \quad x \in (0, 1), \quad \alpha > 0, \quad \beta > 0, \quad q \in (1, \infty)$$

achieves its minimum

$$\left(\alpha^{\frac{1}{1-q}} + \beta^{\frac{1}{1-q}} \right)^{1-q} \quad (\text{resp., } \alpha \wedge \beta \text{ in the case of } q = 1)$$

at

$$x^* = \left[1 + \left(\frac{\alpha}{\beta} \right)^{\frac{1}{q-1}} \right]^{-1} = \beta^{\frac{1}{q-1}} \left[\alpha^{\frac{1}{q-1}} + \beta^{\frac{1}{q-1}} \right]^{-1} \in (0, 1).$$

Applying this result with

$$\alpha = \mu[-M, m], \quad \beta = \mu[n, N], \quad x = \hat{\nu}[m, \bar{\theta}] / \hat{\nu}[m, n]$$

to (20), we get

$$\left(\int_{-M}^N |f - \pi(f)|^q d\mu \right)^{1/q} \geq \left\{ \mu[-M, m]^{\frac{1}{1-q}} + \mu[n, N]^{\frac{1}{1-q}} \right\}^{\frac{1-q}{q}} \hat{\nu}[m, n].$$

Because

$$A \geq \left(\int_{-M}^N |f - \pi(f)|^q d\mu \right)^{1/q} \left(\int_{-M}^N |f'|^p d\nu \right)^{-1/p},$$

the estimate given in the lemma now follows immediately. \square

Now, one may ask the possibility using the idea in the last part of the proof above to improve the estimate produced by proof (a) of Proposition 1.1. The answer is yes if $p > q$ and no if $p \leq q$. Note that here we have power $q > 1$ and in proof (a) of Proposition 1.1, the power is p/q . Thus, if $p > q$, we can follow the proof here to have an improvement. However, in this paper, we are mainly interested in the case that $p \leq q$. Then the function $\alpha x^\gamma + \beta(1-x)^\gamma$ ($\gamma \leq 1$) is concave, its minimum is achieved at the boundaries: either at $x = 0$ or at $x = 1$. That is, $\min_{x \in (0,1)} \{\alpha x^\gamma + \beta(1-x)^\gamma\} = \alpha \wedge \beta$. In this case, we have thus returned to the original result given in proof (a) of Proposition 1.1. We mention that this remark is also meaningful for the first step of the proof of Lemma 2.2 given right below the proof of Lemma 2.1.

As an analog of Lemma 1.5, we have the following result.

Lemma 2.5 Let $q \geq p$. Then we have $B_* \leq B^* \leq 2^{1/p-1/q} B_*$, where the constant B^* is defined by

$$B^* \frac{p}{1-p} = \inf_{x < y} \left\{ \mu[-M, x]^{\frac{p}{(1-p)q}} + \mu[y, N]^{\frac{p}{(1-p)q}} \right\} \hat{\nu}[x, y]^{-1}. \quad (21)$$

Proof. Applying \mathbb{B} to $L^{\frac{q}{q-2}}(\mu)$, the constant $B_{\mathbb{B}}^*$ given in Lemma 2.3 is reduced to B^* defined by (21).

(a) Part $B^* \geq B_*$ follows from the c_r -inequality by setting

$$\alpha = \mu[-M, x]^{\frac{p}{(1-p)q}}, \quad \beta = \mu[y, N]^{\frac{p}{(1-p)q}}, \quad r = \frac{(p-1)q}{p(q-1)} \in (0, 1].$$

(b) Part $B_* \geq 2^{1/q-1/p} B^*$ follows from the inequality by setting

$$\alpha = \mu[-M, x]^{\frac{1}{1-q}}, \quad \beta = \mu[y, N]^{\frac{1}{1-q}}, \quad r = \frac{p(q-1)}{(p-1)q} \geq 1. \quad \square$$

As a combination of Lemmas 2.3 – 2.5, we obtain the following result.

Theorem 2.6 Let $\mu[-M, N] < \infty$ and $\mu_{pp} = 0$. Then

(1) for $1 < p \leq 2 \leq q < \infty$, the optimal constant A in (16) satisfies

$$A \leq k_{2,p} B^*, \text{ where } k_{q,p} \text{ is defined by (5), and}$$

(2) for $1 < p, q < \infty$, we have $A \geq B_*$,

where B_* and B^* are defined in Lemmas 2.4 and (21), respectively. Moreover, we have $B_* \leq B^* \leq 2^{1/p-1/q} B_*$ once $q \geq p$.

Proof. As an application of Lemma 2.3, we get the upper estimate of A . The lower estimate of A is due to Lemma 2.4. The comparison of B^* and B_* comes from Lemma 2.5. \square

When $p = q = 2$, from Theorem 2.6, it follows that

$$B^* \leq A \leq 2B^*.$$

We have thus returned to [6; Theorem 10.2]. It is interesting that in the special case of $p = q = 2$, the duality given in [12; page 13 and (1.17)] coincides with that used in [6, 7]. The former duality exchanges the (single-side but not bilateral) boundary conditions $f(-M) = 0$ and $f(N) = 0$. This is clearly different from a dual of (1) and (16). To prove the last duality, in [6, 7], several techniques were adopted: coupling, duality, and capacity. Thus, the proofs given here are essentially different from that presented in [6, 7], much direct and elementary. Besides, it is unclear how these advanced techniques can be applied to the present setup.

In parallel to the last section, define $y(x)$ to be the solution to the equation $\mu[-M, x] = \mu[y, N]$ and denote by $m(\mu)$ to be the median of μ . Set

$$H_{\mu,\nu}(x, y) = \left[\mu[-M, x]^{\frac{1}{1-q}} + \mu[y, N]^{\frac{1}{1-q}} \right]^{\frac{1-q}{q}} \hat{\nu}[x, y]^{\frac{p-1}{p}}.$$

Define

$$H^o = 2^{1/q-1} \sup_{x \in (-M, m(\mu))} \mu[-M, x]^{1/q} \hat{\nu}[x, y(x)]^{(p-1)/p} \quad (22)$$

Denote by Γ be the limiting points of $H_{\mu,\nu}(x, y)$ as

$$\hat{\nu}[-M, x] = \infty \quad \text{or} \quad \hat{\nu}[y, N] = \infty,$$

as well as the iterated limits if $\hat{\nu}[-M, N] = \infty$ when $M = \infty = N$. Set

$$H^\partial = \begin{cases} \sup\{\gamma : \gamma \in \Gamma\} & \text{if } \Gamma \neq \emptyset \\ 0 & \text{if } \Gamma = \emptyset. \end{cases} \quad (23)$$

Similar to Lemmas 1.7 and 1.8, we have the following result.

Lemma 2.7 Let $\mu[-M, N] < \infty$ and $\mu_{pp} = 0$. Define H^o and H^∂ as above. Then we have

$$H^o \vee H^\partial \leq B_* \leq (2^{1-1/q} H^o) \vee H^\partial$$

and

$$(2^{1/p-1/q} H^o) \vee H^\partial \leq B^* \leq (2^{1-1/q} H^o) \vee H^\partial.$$

Proof. Consider B_* for instance. Recalling that $\mu[-M, x] = \mu[y(x), N]$, we have

$$\begin{aligned} & \sup_{-M \leq x < y \leq N} \left\{ \mu[-M, x]^{\frac{1}{1-q}} + \mu[y, N]^{\frac{1}{1-q}} \right\}^{\frac{1-q}{q}} \hat{\nu}[x, y]^{\frac{p-1}{p}} \\ & \geq 2^{1/q-1} \sup_{-M \leq x \leq m(\mu)} \mu[-M, x]^{1/q} \hat{\nu}[x, y(x)]^{(p-1)/p} \\ & = H^o. \end{aligned}$$

This plus the boundary condition gives us the lower estimate of B_* . The proofs for the other assertions are similar. \square

Corollary 2.8 The Hardy-type inequality (16) holds iff $H^o \vee H^\partial < \infty$.

To generalize Theorem 2.6 to a general normed linear space \mathbb{B} , as in the study of the upper estimate $B_{\mathbb{B}}^*$, a natural way is starting from B_{g*} :

$$\sup_g B_{g*} = \sup_g \sup_{x < y} \left\{ \left(\mu_g[-M, x]^{\frac{1}{1-q}} + \mu_g[y, N]^{\frac{1}{1-q}} \right)^{\frac{1-q}{q}} \hat{\nu}[x, y]^{\frac{p-1}{p}} \right\}.$$

Then it is not clear how to handle with this expression in terms of the norm \mathbb{B} . A crucial point here is that the measure μ appears in the last expression twice rather than a single term in the last section. The next result is an extension and improvement of the basic estimates given in [4; Theorem 2.2].

Theorem 2.9 Let $\mu[-M, N] < \infty$, $\mu_{pp} = 0$, and \mathcal{G} satisfy Hypotheses 1.10. Then

- (1) for $p \in (1, 2]$, the optimal constant $A_{\mathbb{B}}$ in the inequality

$$\|(f - \pi(f))^2\|_{\mathbb{B}}^{1/2} \leq A_{\mathbb{B}} \|f'\|_{\nu, p}$$

satisfies $A_{\mathbb{B}} \leq k_{2,p} B_{\mathbb{B}}^*$, where the constant $B_{\mathbb{B}}^*$ is defined by

$$B_{\mathbb{B}}^{*- \frac{p}{p-1}} = \inf_{-M < x < y < N} \left\{ \|\mathbb{1}_{[-M, x]}\|_{\mathbb{B}}^{-\frac{p}{2(p-1)}} + \|\mathbb{1}_{[y, N]}\|_{\mathbb{B}}^{-\frac{p}{2(p-1)}} \right\} \hat{\nu}[x, y]^{-1}.$$

- (2) For $1 < p, q < \infty$, the optimal constant $A_{\mathbb{B}}$ in the inequality

$$\| |f - \pi(f)|^q \|_{\mathbb{B}}^{1/q} \leq A_{\mathbb{B}} \|f'\|_{\nu, p}$$

satisfies $A_{\mathbb{B}} \geq B_{\mathbb{B}*}$, where

$$B_{\mathbb{B}*} = \sup_{-M < x < y < N} \gamma_{\mathbb{B}}(x, y; q) \hat{\nu}[x, y]^{\frac{p-1}{p}}$$

and

$$\gamma_{\mathbb{B}}(x, y; q) = \inf_{z \in (0, 1)} \|\mathbb{1}_{[-M, x]} z^q + \mathbb{1}_{[y, N]} (1-z)^q\|_{\mathbb{B}}^{1/q}.$$

Proof. The first assertion is a copy of Lemma 2.3. To prove the second assertion, we return to the construction used in the proof of Lemma 2.4. That is, we use the notation $\bar{\theta}$ and f introduced there. First, we have

$$\begin{aligned} \| |f - \pi(f)|^q \|_{\mathbb{B}} &= \| |f|^q \|_{\mathbb{B}} \\ &\geq \| |f|^q \mathbb{1}_{[-M, m]} + |f|^q \mathbb{1}_{[n, N]} \|_{\mathbb{B}} \\ &= \| \mathbb{1}_{[-M, m]} \hat{\nu}[m, \bar{\theta}]^q + \mathbb{1}_{[n, N]} \hat{\nu}[\bar{\theta}, n]^q \|_{\mathbb{B}} \\ &\geq \gamma_{\mathbb{B}}(m, n; q)^q \hat{\nu}[m, n]^q. \end{aligned}$$

Combining this with (19), we obtain

$$\frac{\| |f - \pi(f)|^q \|_{\mathbb{B}}^{1/q}}{\| f' \|_{\nu, p}} \geq \gamma_{\mathbb{B}}(m, n; q) \hat{\nu}[m, n]^{\frac{p-1}{p}}.$$

Now the required assertion follows by making supremum with respect to (x, y) with $x < y$. \square

Example 2.10 Let $\mu(dx) = \nu(dx) = e^{-bx} dx$ ($b > 0$) on $(0, \infty)$. Then the inequality (16) does not hold if $q > p$. When $q = p$, we have

$$B^* = B_* = H^\partial = \frac{1}{b}(p-1)^{1-\frac{1}{p}}.$$

The upper estimate $2B^*$ in Theorem 2.6(1) is sharp in the case of $p = q = 2$, refer to [7; Example 5.3].

Proof. We have

$$\mu(0, x) = \frac{1}{b}(1 - e^{-bx}), \quad \mu(y, \infty) = \frac{1}{b}e^{-by}, \quad y(x) = -\frac{1}{b}\log(1 - e^{-bx}), \quad m(\mu) = \frac{1}{b}\log 2.$$

Next,

$$h(x) = e^{\frac{bx}{p-1}}, \quad \hat{\nu}[x, y] = \frac{p-1}{b} \left(e^{\frac{by}{p-1}} - e^{\frac{bx}{p-1}} \right).$$

Thus,

$$H_{\mu, \nu}(x, y) = b^{-\frac{1}{q}} \left(\frac{p-1}{b} \right)^{1-\frac{1}{p}} \left[(1 - e^{-bx})^{\frac{1}{1-q}} + e^{-\frac{by}{1-q}} \right]^{\frac{1}{q}-1} \left(e^{\frac{by}{p-1}} - e^{\frac{bx}{p-1}} \right)^{1-\frac{1}{p}}.$$

Hence

$$\begin{aligned} H^\partial &= \lim_{y \rightarrow \infty} H_{\mu, \nu}(x, y) = b^{-\frac{1}{q}} \left(\frac{p-1}{b} \right)^{1-\frac{1}{p}} \lim_{y \rightarrow \infty} e^{by \left(\frac{1}{p} - \frac{1}{q} \right)} \\ &= \begin{cases} b^{-1}(p-1)^{1-\frac{1}{p}} & \text{if } q = p \\ \infty & \text{if } q > p. \end{cases} \end{aligned}$$

Thus, the inequality (16) does not hold if $q > p$. Next, assume that $q = p$. Then we have

$$\begin{aligned} H^o &= 2^{\frac{1}{p}-1} b^{-\frac{1}{p}} \left(\frac{p-1}{b} \right)^{1-\frac{1}{p}} \sup_{x \in (0, b^{-1} \log 2)} (1 - e^{-bx})^{\frac{1}{p}} \left((1 - e^{-bx})^{-\frac{1}{p-1}} - e^{\frac{bx}{p-1}} \right)^{1-\frac{1}{p}} \\ &= 2^{\frac{1}{p}-1} b^{-\frac{1}{p}} \left(\frac{p-1}{b} \right)^{1-\frac{1}{p}} \sup_{x \in (0, \log 2)} (1 - e^{-x})^{\frac{1}{p}} \left((1 - e^{-x})^{-\frac{1}{p-1}} - e^{\frac{x}{p-1}} \right)^{1-\frac{1}{p}}. \end{aligned}$$

To compute H^o , we observe that

$$\lim_{x \rightarrow 0} (1 - e^{-x})^{\frac{1}{p}} \left((1 - e^{-x})^{-\frac{1}{p-1}} - e^{\frac{x}{p-1}} \right)^{1-\frac{1}{p}} = 1.$$

Because of this and the decreasing property of the function on the left-hand side, we obtain

$$H^o = 2^{\frac{1}{p}-1} b^{-\frac{1}{p}} \left(\frac{p-1}{b} \right)^{1-\frac{1}{p}} = 2^{\frac{1}{p}-1} \frac{1}{b} (p-1)^{1-\frac{1}{p}}.$$

Having H^∂ and H^o at hand, it is easy to compute B^* and B_* . \square

Example 2.11 Let $\mu(dx) = x^{-2}dx$ and $\nu(dx) = dx$ on $(1, \infty)$. Then the inequality (16) does not hold if $\frac{1}{p} + \frac{1}{q} < 1$. Otherwise, if $\frac{1}{p} + \frac{1}{q} = 1$, then $B^* = B_* = H^\partial = 1$. In particular, when $p = q = 2$, our upper estimate $2B^*$ in Theorem 2.6(1) is exact since $A = 2$ (cf. [7; Example 5.4]). If $\frac{1}{p} + \frac{1}{q} > 1$, then $H^o \leq B_* \leq 2^{1-1/q} H^o$ and $2^{1/p-1/q} H^o \leq B^* \leq 2^{1-1/q} H^o$:

$$H^o = 2^{\frac{1}{q}-1} \left[\frac{2\beta}{\sqrt{\alpha^2 - 6\alpha\beta + \beta^2} + \alpha - \beta} + 1 \right]^\alpha \left[2 - \frac{\sqrt{\alpha^2 - 6\alpha\beta + \beta^2} + \alpha + \beta}{2\beta} \right]^\beta,$$

where $\alpha = 1 - \frac{1}{p} - \frac{1}{q} < 0$ and $\beta = 1 - \frac{1}{p} > 0$. In this case, $2^{1/p-1} \leq B^*/B_* \leq 2^{1-1/q}$. For fixed $p = 5/4$, when q varies over $[2, 4.25]$, the curves of H^o , B_* , B^* , and $2^{3/2-1/p} B^*$ are given in Figure 2. The ratio of the upper and lower bounds is increasing in q but no more than 2.

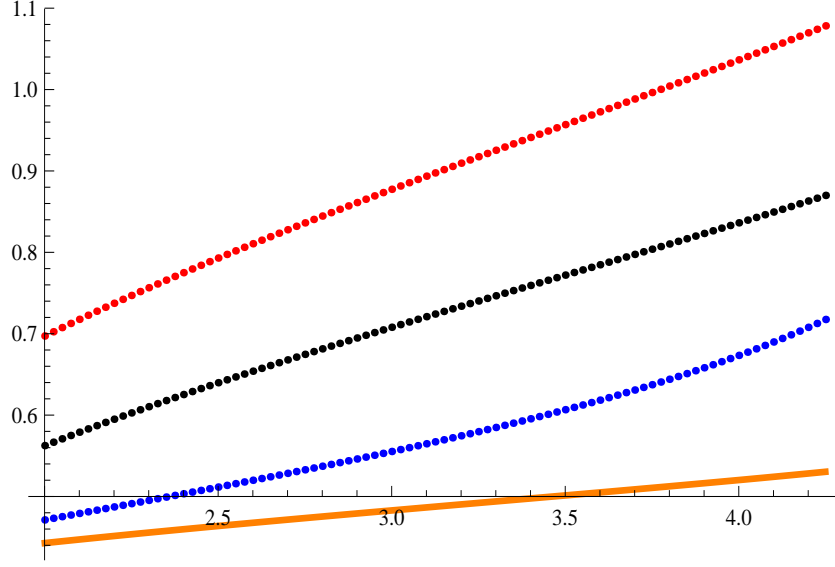


Figure 2 The curves from bottom to top are H^o , B_* , B^* , and $k_{2,p}B^*$, respectively.

Proof. Note that $h \equiv 1$. We have

$$\mu(1, x) = \int_1^x \frac{1}{z^2} = \frac{x-1}{x}, \quad \mu(y, \infty) = \int_y^\infty \frac{1}{z^2} = \frac{1}{y}, \quad y(x) = \frac{x}{x-1}, \quad m(\mu) = 2.$$

Then

$$H_{\mu, \nu}(x, y) = \left[\left(\frac{x-1}{x} \right)^{1/(1-q)} + y^{1/(q-1)} \right]^{1/q-1} (y-x)^{(p-1)/p}.$$

Thus,

$$H^\partial = \overline{\lim}_{y \rightarrow \infty} H_{\mu, \nu}(x, y) = \begin{cases} 1 & \text{if } \frac{1}{p} + \frac{1}{q} = 1 \\ \infty & \text{if } \frac{1}{p} + \frac{1}{q} < 1 \\ 0 & \text{if } \frac{1}{p} + \frac{1}{q} > 1. \end{cases}$$

Hence the inequality does not hold if $\frac{1}{p} + \frac{1}{q} < 1$. When $p = 2$, this means that the inequality does not hold whenever $q > 2$. The assertion is known as a sharp result for Nash inequality (cf. [5; Table 8.2]). Next, we have

$$\begin{aligned} H^o &= 2^{1/q-1} \sup_{x \in (1, 2]} \left(\frac{x-1}{x} \right)^{1/q} \left(\frac{x}{x-1} - x \right)^{(p-1)/p} \\ &= 2^{1/q-1} \sup_{x \in (1, 2]} \left(1 + \frac{1}{x-1} \right)^{1-\frac{1}{p}-\frac{1}{q}} (2-x)^{1-\frac{1}{p}}. \end{aligned}$$

When $\frac{1}{p} + \frac{1}{q} = 1$, we have $H^o = 2^{1/q-1}$, and so $B^* = B_* = H^\partial = 1$. When $\frac{1}{p} + \frac{1}{q} > 1$, we have

$$B^* = \sup_{x < y} \left[\left(1 - \frac{1}{x} \right)^{\frac{p}{(1-p)q}} + y^{\frac{p}{(p-1)q}} \right]^{\frac{1}{p}-1} (y-x)^{1-\frac{1}{p}},$$

$$B_* = \sup_{x < y} \left[\left(1 - \frac{1}{x} \right)^{\frac{1}{1-q}} + y^{\frac{1}{q-1}} \right]^{\frac{1}{q}-1} (y-x)^{1-\frac{1}{p}}.$$

H^o has an explicit expression as shown above. \square

3 Nash inequality, Sobolev-type inequality, and logarithmic Sobolev inequality

In this section, we study first the Nash inequality and its closely related Sobolev-type inequality, as a typical application of Theorem 2.6. Then we study the logarithmic Sobolev inequality by the similar method introduced in the paper.

Nash inequality and Sobolev-type inequality

Recall that the probability measure π is defined by $\mu/\mu[-M, N]$. Consider the Nash inequality:

$$\|f - \pi(f)\|_{\mu, 2}^{2+4/\gamma} \leq A_N \|f'\|_{\nu, 2}^2 \|f\|_{\mu, 1}^{4/\gamma}, \quad \gamma > 0. \quad (24)$$

It seems more symmetric to replace $\|f\|_{\mu, 1}$ by $\|f - \pi(f)\|_{\mu, 1}$ on the right-hand side of (24). Let us denote the latter one by (24)'. If (24) holds for every absolutely continuous $f \in L^2(\mu)$, then so does (24)' regarding $f - \pi(f)$ as a new f . Conversely, assume (24)'. Since $\|f - \pi(f)\|_{\mu, 1} \leq 2\|f\|_{\mu, 1}$, we certainly have (24). Hence (24) and (24)' are equivalent.

It is known (cf. [15], [5; §4.8 and §6.5] for related results and more references) that this inequality, when $\gamma > 2$, is equivalent to the Sobolev-type inequality

$$\|f - \pi(f)\|_{\mu, 2\gamma/(\gamma-2)}^2 \leq A_S \|f'\|_{\nu, 2}^2. \quad (25)$$

Now, as an application of Theorem 2.6, we have the following result.

Theorem 3.1 Let $\mu[-M, N] < \infty$ and $\mu_{pp} = 0$. Then

- (1) when $\gamma > 2$, the Nash inequality (24) or equivalently, the Sobolev-type inequality (25) holds iff $H^o \vee H^\partial < \infty$, where H^o and H^∂ are defined by (22) and (23), respectively, with $p = 2$ and $q = 2\gamma/(\gamma - 2)$. Furthermore, we have

$$B_{S*} \leq A_S \leq 4B_S^*,$$

where

$$B_S^* = \sup_{x < y} \left[\mu(-M, x)^{\frac{2}{\gamma}-1} + \mu(y, N)^{\frac{2}{\gamma}-1} \right]^{-1} \hat{\nu}[x, y],$$

$$B_{S*} = \inf_{x < y} \left[\mu(-M, x)^{\frac{4}{2+\gamma}-1} + \mu(y, N)^{\frac{4}{2+\gamma}-1} \right]^{-\frac{2}{\gamma}-1} \hat{\nu}[x, y],$$

and $B_{S*} \leq B_S^* \leq 2^{2/\gamma} B_{S*}$.

(2) Let $M, N = \infty$. If $\hat{\nu}(-\infty, \theta] \wedge \hat{\nu}[\theta, \infty) = \infty$,

$$\overline{\lim}_{x \rightarrow -\infty} \mu(-\infty, x) \hat{\nu}(x, \theta) < \infty \quad \text{and} \quad \overline{\lim}_{y \rightarrow \infty} \mu(y, \infty) \hat{\nu}(\theta, y) < \infty,$$

$$\underline{\lim}_{x \rightarrow -\infty} \mu(-\infty, x) \hat{\nu}(x, \theta) > 0 \quad \text{and} \quad \underline{\lim}_{y \rightarrow \infty} \mu(y, \infty) \hat{\nu}(\theta, y) > 0$$

for some $\theta \in \mathbb{R}$, then the Nash inequality (24) does not hold whenever $\gamma \in (0, 2]$.

Proof. (a) The assertion in Part (1) on the estimates of A_S is a straightforward consequence of Theorem 2.6 with $p = 2$, $q = 2\gamma/(\gamma - 2)$, and $(A_S, B_S^*, B_{S*}) = (A^2, B^{*2}, B_*^2)$. The criterion is a copy of Corollary 2.8.

(b) The restriction “ $\gamma > 2$ ” comes from the reduction of Nash inequality to the Sobolev-type one. This costs a smaller gap of the criterion for the Nash inequality, marked as (ε) in the last line on page 15 and the last sentence in Theorem 1.10 of [5], for instance. The restriction is recently removed in [19] in the discrete situation. Here we show that a direct proof is also possible under a technical condition.

To see that the Nash inequality (24) does not hold for $\gamma \in (1, 2]$, rewrite for a moment the inequality as

$$\text{Var}_\mu(f)^r \leq A_r \|f'\|_{\nu, 2}^2, \quad f \in L^2(\mu), \quad \|f\|_{\mu, 1} = 1, \quad (26)$$

where A_r denotes the optimal constant. When γ varies 0 to 2, r moves from ∞ to 2. By the splitting technique (replacing $\|f\|_{\mu, 1}$ by $\|f - \pi(f)\|_{\mu, 1}$ on the right-hand side of (24)), we may consider the half-space $(-M, N) = (0, \infty)$ only, and reduce (26) to

$$\|f\|_{\mu, 2}^{2r} \leq C_r \|f'\|_{\nu, 2}^2, \quad f(0) = 0, \quad f \in L^2(\mu), \quad \|f\|_{\mu, 1} = 1,$$

Since $\|f\|_{\mu, 2} \geq \|f\|_{\mu, 1} = 1$, it is clear that the last inequality becomes stronger when r increases. Thus, it is sufficient to show that the inequality (26) does not hold when $r = 2$ (i.e., $\gamma = 2$).

To do so, fix a point $y > 0$ and let

$$f(x) = \hat{\nu}[0, x \wedge y], \quad x \geq 0.$$

Then

$$\begin{aligned}\|f\|_{\mu,1} &= \int_0^\infty \mu(dz) \hat{\nu}[0, z \wedge y] = \int_0^y h \mu(\cdot, \infty), \\ \|f\|_{\mu,2}^{2+4/\gamma} &\geq \mu(y, \infty)^{1+2/\gamma} \hat{\nu}[0, y]^{2+4/\gamma}, \\ \|f'\|_{\nu,2}^2 &= \hat{\nu}[0, y].\end{aligned}$$

Now, we have

$$\begin{aligned}\frac{\|f\|_{\mu,2}^{2+4/\gamma}}{\|f'\|_{\nu,2}^2 \|f\|_{\mu,1}^{4/\gamma}} &\geq \frac{(\mu(y, \infty) \hat{\nu}[0, y]^2)^{1+2/\gamma}}{\hat{\nu}[0, y] [\int_0^y h \mu(\cdot, \infty)]^{4/\gamma}} \\ &= \frac{\mu(y, \infty)^{1+2/\gamma} \hat{\nu}[0, y]^{1+4/\gamma}}{[\int_0^y h \mu(\cdot, \infty)]^{4/\gamma}} \\ &= \frac{\mu(y, \infty)^{1-2/\gamma} \hat{\nu}[0, y]}{[\int_0^y h \mu(\cdot, \infty) / (\mu(y, \infty) \hat{\nu}[0, y]) + 1]^{4/\gamma}}.\end{aligned}$$

Since $\int_0^y h \mu(\cdot, \infty) > \mu(y, \infty) \hat{\nu}[0, y]$, when $\gamma = 2$, we need only to study the ratio

$$\hat{\nu}[0, y] \left[\frac{\mu(y, \infty) \hat{\nu}[0, y]}{\int_0^y h \mu(\cdot, \infty)} \right]^2.$$

By assumption, if $\int_0^\infty h \mu(\cdot, \infty) < \infty$, then the right-hand side goes to infinity as so does y by assumption again. This implies that $A_N = \infty$. Therefore the Nash inequality (24) does not hold at $\gamma \in (0, 2]$ in this case.

Next, if $\int_0^\infty h \mu(\cdot, \infty) = \infty$, then

$$\begin{aligned}\frac{\hat{\nu}[0, y]}{[\int_0^y h \mu(\cdot, \infty)]^2} &\sim \frac{h(y)}{h(y) \mu(y, \infty) \int_0^y h \mu(\cdot, \infty)} \quad (\text{by l'Hôpital's rule}) \\ &= \frac{\hat{\nu}[0, y]}{\mu(y, \infty) \hat{\nu}[0, y] \int_0^y h \mu(\cdot, \infty)} \\ &\sim \frac{\hat{\nu}[0, y]}{\int_0^y h \mu(\cdot, \infty)} \\ &\sim \frac{h(y)}{h(y) \mu(y, \infty)} \\ &= \frac{1}{\mu(y, \infty)} \rightarrow \infty \quad \text{as } y \rightarrow \infty.\end{aligned}$$

Therefore we have arrived the required assertion again. \square

We remark that the condition

$$\overline{\lim}_{x \rightarrow -\infty} \mu(-\infty, x) \hat{\nu}(x, \theta) < \infty \quad \text{and} \quad \overline{\lim}_{y \rightarrow \infty} \mu(y, \infty) \hat{\nu}(\theta, y) < \infty$$

in Theorem 3.1 (2) means that the corresponding diffusion process is exponentially ergodic, otherwise, the Nash inequality can not hold since the latter

inequality is stronger than the former ergodicity (cf. [5; Table 5.1 and Theorem 1.9]). Actually, by the cited results, once the transition probability of the process has a density, one can even assume a stronger condition that $\int_0^\infty h \mu(\cdot, \infty) < \infty$, then the proof above can be simplified.

The rough factor $4^{1+1/\gamma}$ ($\gamma > 2$) in Theorem 3.1 (1) is clearly smaller than the first factor 8 given by [5; Theorem 6.8]. The second factor given by the cited theorem is clearly less sharp than the first one and so is than what we have here.

Logarithmic Sobolev inequality

We now turn to study the logarithmic Sobolev inequality

$$\text{Ent}_\pi(f^2) \leq A_{LS} \|f'\|_{\nu, 2}^2, \quad (27)$$

where

$$\text{Ent}_\pi(f) = \int_{-M}^N f \log \left(\frac{f}{\pi(f)} \right) d\pi \quad \text{for } f \geq 0.$$

Theorem 3.2 Let $\mu[-M, N] < \infty$ and $\mu_{pp} = 0$. Then the optimal constant A_{LS} in (27) satisfies the following estimates

$$B_* \leq A_{LS} \leq 4B^*,$$

where

$$\begin{aligned} B^{*-1} &= \inf_{x < y} \left\{ \hat{\nu}[x, y]^{-1} \left(\left[\pi[-M, x] \log \left(1 + \frac{e^2}{\pi[-M, x]} \right) \right]^{-1} \right. \right. \\ &\quad \left. \left. + \left[\pi[y, N] \log \left(1 + \frac{e^2}{\pi[y, N]} \right) \right]^{-1} \right) \right\}, \\ B_*^{-1} &= \inf_{\theta \in (-M, N)} \inf_{(x, y) \ni \theta} \left\{ \hat{\nu}[x, y]^{-1} \left(\left[\pi[-M, x] \log \left(1 + \frac{1 - \pi[-M, \theta]}{\pi[-M, x]} \right) \right]^{-1} \right. \right. \\ &\quad \left. \left. + \left[\pi[y, N] \log \left(1 + \frac{1 - \pi[\theta, N]}{\pi[y, N]} \right) \right]^{-1} \right) \right\}, \end{aligned}$$

or alternatively,

$$\begin{aligned} B_*^{-1} &= \inf_{x < y} \left\{ \hat{\nu}[x, y]^{-1} \left(\left[\pi[-M, x] \log \left(1 + \frac{z^*(x, y)}{\pi[-M, x]} \right) \right]^{-1} \right. \right. \\ &\quad \left. \left. + \left[\pi[y, N] \log \left(1 + \frac{1 - z^*(x, y)}{\pi[y, N]} \right) \right]^{-1} \right) \right\}, \quad (28) \end{aligned}$$

where $z^*(x, y)$ is the unique solution to the equation

$$\left[\pi[-M, x] \log \left(1 + \frac{z}{\pi[-M, x]} \right) \right]^2 \left(1 + \frac{z}{\pi[-M, x]} \right)$$

$$= \left[\pi[y, N] \log \left(1 + \frac{1-z}{\pi[y, N]} \right) \right]^2 \left(1 + \frac{1-z}{\pi[y, N]} \right), \quad z \in (0, 1). \quad (29)$$

In particular, we have

$$B_*^{-1} \leq \inf_{(x, y) \ni m(\pi)} \left\{ \hat{\nu}[x, y]^{-1} \left(\left[\pi[-M, x] \log \left(1 + \frac{1}{2\pi[-M, x]} \right) \right]^{-1} + \left[\pi[y, N] \log \left(1 + \frac{1}{2\pi[y, N]} \right) \right]^{-1} \right) \right\},$$

where $m(\pi)$ is the median of π .

Proof. (a) Upper bound. Even though this theorem is not a consequence of Theorem 2.9, the idea of the proof for the upper bound is more or less the same as we used several times in the last two sections. Given $f \in L^2(\pi)$ with $\nu(f'^2) \in (0, \infty)$ and $\theta \in (-M, N)$, let

$$\tilde{f} = f - f(\theta), \quad \tilde{f}_- = \tilde{f} \mathbb{1}_{[-M, \theta]}, \quad \tilde{f}_+ = \tilde{f} \mathbb{1}_{(\theta, N]}.$$

The following facts were proved in the first part of [1; Proof of Theorem 3] for the specific $\theta = m(\pi)$, but the proof remains true for general θ :

$$\begin{aligned} \text{Ent}_\pi(f^2) &\leq \text{Ent}_\pi(\tilde{f}^2) + 2\pi(\tilde{f}^2) \quad (\text{by [5; Lemma 4.14]}) \\ &\leq \text{Ent}_\pi(\tilde{f}_-^2) + 2\pi(\tilde{f}_-^2) + \text{Ent}_\pi(\tilde{f}_+^2) + 2\pi(\tilde{f}_+^2) \end{aligned}$$

(since Ent_π is sub-additive: $\text{Ent}_\pi(f + g) \leq \text{Ent}_\pi(f) + \text{Ent}_\pi(g)$) and

$$\text{Ent}_\pi(\tilde{f}_\pm^2) + 2\pi(\tilde{f}_\pm^2) \leq 4B_\theta^\pm \nu(\tilde{f}'^2),$$

where

$$\begin{aligned} B_\theta^- &= \sup_{x < \theta} \pi[-M, x] \log \left(1 + \frac{e^2}{\pi[-M, x]} \right) \hat{\nu}[x, \theta], \\ B_\theta^+ &= \sup_{y > \theta} \pi[y, N] \log \left(1 + \frac{e^2}{\pi[y, N]} \right) \hat{\nu}[\theta, y]. \end{aligned}$$

Thus, we have

$$\begin{aligned} \frac{\text{Ent}_\pi(f^2)}{\nu(f'^2)} &\leq \frac{\text{Ent}_\pi(\tilde{f}^2) + 2\pi(\tilde{f}^2)}{\nu(f'^2)} \\ &\leq \frac{\text{Ent}_\pi(\tilde{f}_-^2) + 2\pi(\tilde{f}_-^2) + \text{Ent}_\pi(\tilde{f}_+^2) + 2\pi(\tilde{f}_+^2)}{\nu(f'^2) + \nu(f'^2)} \\ &\leq \frac{\text{Ent}_\pi(\tilde{f}_-^2) + 2\pi(\tilde{f}_-^2)}{\nu(f'^2)} \vee \frac{\text{Ent}_\pi(\tilde{f}_+^2) + 2\pi(\tilde{f}_+^2)}{\nu(f'^2)} \\ &\leq (4B_\theta^-) \vee (4B_\theta^+). \end{aligned}$$

The original proof for the upper estimate stopped here with $\theta = m(\pi)$. Because θ is arbitrary, we obtain

$$\frac{\text{Ent}_\pi(f^2)}{\nu(f'^2)} \leq 4 \inf_\theta (B_\theta^- \vee B_\theta^+).$$

Note that the right-hand side is independent of f . By choosing $\bar{\theta}$ such that $B_{\bar{\theta}}^- = B_{\bar{\theta}}^+$, it follows that

$$\frac{\text{Ent}_\pi(f^2)}{\nu(f'^2)} \leq 4B_{\bar{\theta}}^-.$$

Now, since f with $\nu(f'^2) \in (0, \infty)$ is arbitrary, we obtain

$$A_{LS} = \sup_{\nu(f'^2) \in (0, \infty)} \frac{\text{Ent}_\pi(f^2)}{\nu(f'^2)} \leq 4B_{\bar{\theta}}^-.$$

Next, as an application of Lemma 1.2, we have

$$\begin{aligned} B^* &= \sup_{x < y} \frac{\hat{\nu}[x, \bar{\theta}] + \hat{\nu}(\bar{\theta}, y]}{[\pi[-M, x] \log(1 + \frac{e^2}{\pi[-M, x]})]^{-1} + [\pi[y, N] \log(1 + \frac{e^2}{\pi[y, N]})]^{-1}} \\ &\geq B_{\bar{\theta}}^- \wedge B_{\bar{\theta}}^+ \\ &= B_{\bar{\theta}}^-. \end{aligned}$$

Combining the last two estimates together, we obtain the required upper bound. Once again, the unknown $\bar{\theta}$ disappears in the expression of B^* .

(b) Lower bound. We adopt a similar method as used in the proof of Lemma 2.4. Define

$$f(z) = -\mathbb{1}_{\{z \leq \theta\}} \hat{\nu}[x \vee z, \theta] + \mathbb{1}_{\{z > \theta\}} \hat{\nu}[\theta, y \wedge z], \quad z \in [-M, N],$$

where x, y, θ with $(x, y) \ni \theta$ are fixed. First, let us apply [1; Proof of Theorem 3] to this specific test function f .

$$\begin{aligned} \text{Ent}_\pi(f^2 \mathbb{1}_{\{z \leq \theta\}}) &= \sup \left\{ \int_{-M}^{\theta} f^2 g d\pi : \int_{-M}^{\theta} e^g d\pi \leq 1 \right\} \\ &\geq \sup \left\{ \int_{-M}^{\theta} f^2 g d\pi : g \geq 0 \text{ and } \int_{-M}^{\theta} e^g d\pi \leq 1 \right\} \\ &\geq \hat{\nu}[x, \theta]^2 \sup \left\{ \int_{-M}^{\theta} \mathbb{1}_{[-M, x]} g d\pi : g \geq 0 \text{ and } \int_{-M}^{\theta} e^g d\pi \leq 1 \right\}. \end{aligned}$$

Applying [1; Lemma 6] to the last supremum, it follows that

$$\text{Ent}_\pi(f^2 \mathbb{1}_{[-M, \theta]}) \geq \hat{\nu}[x, \theta]^2 \varphi(x, \theta),$$

$$\varphi(x, \theta) := \pi[-M, x] \log \left(1 + \frac{1 - \pi[-M, \theta]}{\pi[-M, x]} \right). \quad (30)$$

Symmetrically, we have

$$\begin{aligned} \text{Ent}_\pi(f^2 \mathbb{1}_{[\theta, N]}) &\geq \hat{\nu}[\theta, y]^2 \psi(\theta, y), \\ \psi(\theta, y) &:= \pi[y, N] \log \left(1 + \frac{1 - \pi[\theta, N]}{\pi[y, N]} \right). \end{aligned} \quad (31)$$

Next, by logarithmic Sobolev inequality,

$$A_{LS} \nu(f'^2 \mathbb{1}_{[-M, \theta]}) \geq \text{Ent}_\pi(f^2 \mathbb{1}_{[-M, \theta]}),$$

it follows that

$$A_{LS} \nu(f'^2 \mathbb{1}_{[-M, \theta]}) \geq \hat{\nu}[x, \theta]^2 \varphi(x, \theta).$$

Similarly,

$$A_{LS} \nu(f'^2 \mathbb{1}_{[\theta, N]}) \geq \hat{\nu}[\theta, y]^2 \psi(\theta, y).$$

We now arrive at the place different from the known proofs. Summing up the last two inequalities, it follows that

$$A_{LS} \nu(f'^2) \geq \hat{\nu}[x, \theta]^2 \varphi(x, \theta) + \hat{\nu}[\theta, y]^2 \psi(\theta, y).$$

Since one can replace $\nu(f'^2)$ by $\hat{\nu}(f'^2)$ in the original inequality, by definition of f , we have $\hat{\nu}(f'^2) = \hat{\nu}[x, y]$, and so

$$A_{LS} \geq \frac{1}{\hat{\nu}[x, y]} \left[\hat{\nu}[x, \theta]^2 \varphi(x, \theta) + \hat{\nu}[\theta, y]^2 \psi(\theta, y) \right].$$

Therefore, we have

$$A_{LS} \geq \hat{\nu}[x, y] \left[\left(\frac{\hat{\nu}[x, \theta]}{\hat{\nu}[x, y]} \right)^2 \varphi(x, \theta) + \left(1 - \frac{\hat{\nu}[x, \theta]}{\hat{\nu}[x, y]} \right)^2 \psi(\theta, y) \right].$$

Noting that the function $c_1 z^2 + c_2(1 - z)^2$ on $[0, 1]$ achieves its minimum $(c_1^{-1} + c_2^{-1})^{-1}$ at $z^* = c_2/(c_1 + c_2)$, it follows that

$$A_{LS} \geq \hat{\nu}[x, y] (\varphi(x, \theta)^{-1} + \psi(\theta, y)^{-1})^{-1}.$$

Since $(x, y) \ni \theta$ are arbitrary, we finally arrive at

$$A_{LS} \geq \sup_{\theta \in (-M, N)} \sup_{(x, y) \ni \theta} \hat{\nu}[x, y] (\varphi(x, \theta)^{-1} + \psi(\theta, y)^{-1})^{-1}. \quad (32)$$

This gives us the first version of B_* . Then the final assertion of the theorem follows by setting $\theta = m(\pi)$.

(c) We now prove the alternative assertion of B_* . Since supremums are exchangeable, one may rewrite B_* as

$$B_* = \sup_{x < y} \left\{ \hat{\nu}[x, y] \sup_{\theta \in (x, y)} (\varphi(x, \theta)^{-1} + \psi(\theta, y)^{-1})^{-1} \right\}.$$

Fix $x < y$ and make a change of the variable θ by $z = 1 - \pi[-M, \theta]$. Then $1 - \pi[\theta, N] = 1 - z$ and the functions $\varphi(x, \theta)$ and $\psi(\theta, y)$ become $\tilde{\varphi}(x, z)$ and $\tilde{\psi}(z, y)$, respectively. The supremum above should be achieved at the point for which the derivative in z of $(\tilde{\varphi}(x, z)^{-1} + \tilde{\psi}(z, y)^{-1})^{-1}$ vanishes. This leads to the unique solution $z^* = z^*(x, y)$ to equation (29). Then we obtain (28). \square

We mention that there is a large number of publications on the logarithmic Sobolev inequalities, in the one-dimensional case for instance, one may refer to [14], [1], [5; §4.6 and §6.6] for related results and more references. Generally speaking, Theorem 3.2 clearly improves [1; Theorem 3] (since $\alpha \vee \beta$ is replaced by $\alpha + \beta$), to which the factor of the upper and lower bounds is at most 16, the best one we have known up to now. In the special case that the measures π and $\hat{\nu}$ are symmetric with respect to $m(\pi)$, the computation of B^* and B_* can be reduced to half space. Then Theorem 3.2 coincides with [1; Theorem 3]. Besides, having Theorem 3.2 at hand, it should be easy to introduce the corresponding H^o and H^∂ as we did in the previous sections.

The methods introduced in the paper should have more applications. For instance, in parallel to the proof of Theorem 3.2, we may have an improved version of the Sobolev inequality and the Łatała–Oleszkiewicz inequality presented by [1; Theorems 11 and 13].

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